Optimal Estimation Over the Collision Channel

Marcos M. Vasconcelos and Nuno C. Martins

***Abstract*—Consider a distributed sensing system that comprises two sensors, each observing a random variable, and a remote estimator. The goal of the remote estimator is to produce estimates of the random variables based on information transmitted to it by the sensors. The random variables are independent and information is transferred from the sensors to the estimator via a collision channel, which can only convey a single packet. Each sensor has the authority to decide what and when to transmit, and simultaneous transmissions result in a collision event to be detected at the estimator. In our formulation, there is no communication between the sensors, which precludes the use of coordinated strategies. Our results characterize the structure of policies at the sensors and the remote estimator that are optimal with respect to a mean squared error criterion. More specifically, we show that there exist optimal policies at the sensors that use deterministic threshold strategies to decide when to transmit. This structural result is independent of the distributions of the observed random variables. In our analysis, we prove that the computation of a person-by-person optimal threshold policy can be recast as a one-bit optimal quantization problem for which the cost is nonuniform across representation symbols. Based on that observation, we provide an iterative procedure akin to the Lloyd–Max algorithm that can be used to compute locally optimal solutions. In the Gaussian case, our iterative method converged to an optimal solution in all numerical examples we have tried. We provide several examples that show the optimality of asymmetric threshold policies even when the overall framework is symmetric, such as when both random variables are Gaussian with zero mean and the same variance.**

***Index Terms*—Decision theory, estimation, multi-agent systems, networked control systems, optimization, quantization.**

I. Introduction

CYBER-PHYSICAL systems are often formed by multiple noncollocated components that sense, exchange information, and act as a team through a network [1]. In the wireless case, the network may support only a finite number of simultaneous transmissions due to limitations such as interference. Here, we are interested in characterizing optimal policies and the performance degradation that occurs when the maximal number of simultaneous transmissions is strictly less than the number of components sharing the same network. In order to obtain design principles that can be characterized analytically, we consider a simple configuration formed by a remote estimator that operates on information received from two sensors that measure a random variable each (see Fig. 1). Each sensor has the authority to decide whether to attempt a transmission or to remain silent based solely on the random variable it measures. In our formulation, we assume that information is conveyed through a *collision channel* for which at most one transmission can reliably reach the estimator and multiple transmissions result in a collision. We consider the design of policies that minimize a mean squared estimation error, subject to the communication constraint imposed by the collision channel. In particular, we will prove the optimality of policies with a particular threshold (event-based) structure and explore iterative methods to obtain person-by-person optimal solutions.

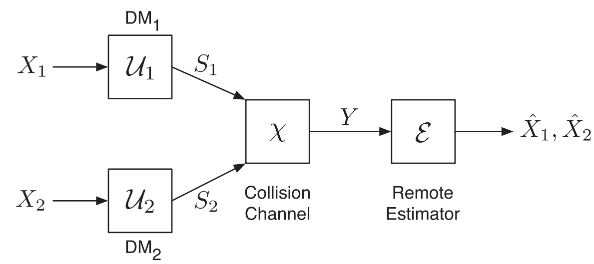

Fig. 1. Schematic representation of distributed estimation over a collision channel.

A. Applications

The framework proposed here can be used to model a distributed sensor network in which measurements that are made by noncollocated sensors are wirelessly transmitted to a fusion center. One-shot problem formulations arise when the objective is to detect a one-time event of interest or estimating vital variables in real time, and with minimal delay. This is the case when multiple sensors monitor large structures or systems such as bridges, electric power grids and, oil and gas pipelines, which are subject to potentially catastrophic events. In such scenarios, the delay and additional infrastructure required for coordinating access to the network over large distances would be costly and impede swift detection and estimation. Here we adopt a one-shot formulation in which each sensor does not have time to coordinate or communicate with the other, and must decide whether to communicate based solely on its measurement. The observations are independent, but otherwise allowed to be arbitrarily distributed, possibly coming from different families of distributions.

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The authors are with the Department of Electrical and Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742 USA (e-mail: marcos@umd.edu; nmartins@umd.edu).

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B. Related Literature

Control and estimation over communication networks have been of great interest to control theorists and engineers during the last decade [2]. With the advent of cyber-physical systems as a new paradigm for system design, the development of new tools and models in networked control and estimation are as important now as ever. The components or blocks of a cyber-physical system are noncollocated and typically interconnected by a network. Moreover, these blocks have access to different information, which is often corrupted by noise, delayed or incomplete due to physical or operational constraints. Such problems can be cast in the framework of team decision theory and their analysis often combine tools from control, information and optimization theories [3].

Many channel and network models have been studied in the existing networked control and estimation literature. Apart from the traditional additive Gaussian noise (AWGN) and fading channels, the packet drop channel (also known as analog erasure channel) has attracted most of the attention of the research community in control. Most notably, the works of Sinopoli *et al.* [4] and Gupta *et al.* [5], [6] have become landmark references in the area. However, there have been only few studies that explicitly deal with the effects of interference in control and estimation over wireless networks [7]–[11]. Our work seeks to contribute in this growing field by modeling interference using a simple model for a communication medium shared by multiple devices known as the *collision channel* [12], which has been largely used, along with queueing theory, in the design and analysis of wireless networks [13].

We assume that the collision channel can only carry one packet and differs from the packet drop channel in the following fundamental aspect: the channel output alphabet has two distinct symbols to represent no-transmission (idle channel) and collision (simultaneous transmissions) events. Therefore, the receiver is always able to detect if the transmitters attempted to communicate or not, even though the colliding packets cannot be correctly decoded. We present a precise mathematical definition for the collision channel that makes the distinction between collision and no-transmission events. Then, we exploit these two symbols to embed an extra bit of information in the communication between sensors and remote estimator. We specifically treat the search for an optimal communication policy as a one-bit quantizer design problem with a new asymmetric quadratic distortion metric.

Our collision channel model can be modified to incorporate features of more sophisticated systems that allow, for instance, asynchronous access [14] and multipacket reception capabilities [15], [16]. There are also variants that assume sequential transmissions with and without feedback [17]. One of the possible extensions of the present framework is the collision channel with capture, where the sensors may also adjust the power used to transmit a packet, and in the event of a collision, the packet transmitted with the largest power survives the collision and the others are lost [18].

Since the communication between the components in a cyber-physical system is usually performed over a network of limited capacity (or limited infrastructure), it is important to understand how these limitations may degrade the performance of the overall system. More importantly, the system's designer must provide strategies that make the best use of the limited available communication resources. There are three main lines of research that consider the effects of communication links between sensors and estimators. The first class of problems corresponds to the characterization of fundamental limitations on performance caused by noisy communication channels [19]. Here we find the more traditional communication models such as the AWGN and fading channels [20]–[22]; as well as the packet drop channel of [4]–[6]. The second class of problems studies the effect of noiseless but rate limited channels in estimation and control, in which signals are quantized prior to transmission. Results known as *data-rate theorems* [23] establish the minimum quantization rate necessary to stabilize an unstable plant, while other works establish conditions for the existence of stable quantizer-estimator schemes [24].

Finally, a third class of problems studies the tradeoff between communication rate and estimation performance over noisy or noiseless but costly communication channels. An interesting common feature of these problems is that threshold policies *emerge* as solutions to optimization problems and are not an architecture *imposed* by the system's designer. The first contributions in this field were done by Imer and Basar in [25], where a limit on the number of noiseless transmissions that can be made by the sensor over a finite horizon imposes an upper bound on the communication rate. The idea that event-based estimation/control systems can be used for *signalling* was first mentioned by [25], whose results were later complemented by [26]. A continuous time formulation of this problem was studied by Rabi *et al.* in [27]. Xu and Hespanha in [28] solved an infinite horizon problem whose objective functional combined the expected estimation error and a communication cost. Lipsa and Martins in [29] also considered a finite horizon problem with an objective functional that combines estimation error and communication costs and established the structure of jointly optimal communication and estimation policies. In particular, [29] shows that there exist optimal communication policies of the symmetric threshold type and the optimal estimator admits a simple recursive structure. In [30], the authors showed that this structure is preserved when the channel randomly drops packets. Nayyar *et al.* in [31] generalized [25] and [29] obtaining structural results for when, in addition to communication costs, there is a stochastically varying energy budget, which reflects the sensor's ability to harvest energy from the environment in order to communicate. In the context of control of dynamical systems over communication networks, the work of Molin and Hirche in [32] shows that certainty equivalence controllers are optimal for point-to-point communication links are of the type in [25] and [29]. A model similar to the one presented here was used Gupta *et al.* in a game between a sensor and a jammer in a remote estimation problem [33].

C. Comparison With Prior Work by the Authors

This paper is an extended version of [34], which first introduced the problem and established early versions of some of the structural results detailed here. The analysis in [34] assumed independent Gaussian measurements, whereas in this paper, we prove a more general version in which X_1 and X_2 are any independent continuous random variables with finite second moment, possibly with different distributions. We answer several questions that were left open in [34], such as the existence of team-optimal solutions and we provide a rigorous proof that the duality gap is zero in one of our key lemmas. Here, we formally derive the Modified Lloyd–Max algorithm and provide a novel nonlinear dynamical system interpretation, for which several properties can be established. Many examples are provided, including pairs of person-by-person optimal solutions for the Gaussian case. Such person-by-person optimal solutions were not available at the time when [34] was written.

D. Paper Organization

This paper is structured in seven sections, including the introduction. The problem formulation and our main result are stated in Section II. The optimality of threshold policies for the system in Fig. 1 is established in Section III, where we use a person-by-person optimality approach to establish an analogy with a framework of remote estimation with communication costs. In Section IV, we show that the computation of person-by-person optimal threshold policies is equivalent to the design of a one-bit quantizer that minimizes a distortion metric that is nonuniform across quantization regions. We argue that asymmetric thresholds are optimal when it is advantageous to encode information using collisions and no-transmissions. The numerical examples discussed in Section V illustrate that the policy design is a nonconvex problem in general, and optimal solutions can be symmetric or asymmetric, depending on the parameters that specify the problem. A numerical approach to compute locally optimal threshold policies is proposed in Section VI based on a modified version of the Lloyd–Max algorithm. We present examples for the case when the variables are Gaussian that illustrate that the algorithm converges to a local minimum[1] and that it can be used to compute person-by-person optimal solutions. The paper ends with our conclusions in Section VII.

E. Notation

We use the terminology *sensor* and *decision maker* (DM) interchangeably throughout the paper. We adopt the following notation: random variables and random vectors are represented using upper case letters, such as X. Realizations of random variables and random vectors are represented by the corresponding lower case letter, such as x. The probability density function of a continuous random variable X, provided that it is well defined, is denoted by f_X. Functions and functionals are denoted using calligraphic letters such as $\mathcal{F}$ or $\mathscr{F}$. We use $\mathcal{B}(p)$ and $\mathcal{N}(m,\sigma^2)$ to represent the Bernoulli probability mass function of parameter p and the Gaussian probability distribution of mean m and variance σ^2, respectively. The real line is denoted by $\mathbb{R}$. Sets are represented in blackboard bold font, such as $\mathbb{A}$. The complement of a subset $\mathbb{A}$ in $\mathbb{R}$ is denoted by $\mathbb{A}^c \stackrel{\text{def}}{=} \mathbb{R}\setminus\mathbb{A}$. The empty set is denoted by $\emptyset$. The extended real line is defined as $\bar{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R}\cup\{-\infty,+\infty\}$. The probability of an event $\mathfrak{E}$ is denoted by $\mathbf{P}(\mathfrak{E})$; the expectation and variance of a random variable Z are denoted by $\mathbf{E}[Z]$ and $\mathbf{V}(Z)$, respectively. The positive and negative parts of a real-valued function $\mathcal{G}$ are defined as $[\mathcal{G}(x)]^+ \stackrel{\text{def}}{=} \max\{0,\mathcal{G}(x)\}$ and $[\mathcal{G}(x)]^- \stackrel{\text{def}}{=} \max\{0,-\mathcal{G}(x)\}$. We denote by $L^p_\mu(\mathbb{R})$ the space of all μ-measurable functions $\mathcal{G}:\mathbb{R}\to\mathbb{R}$ such that $\int_{\mathbb{R}}|\mathcal{G}(x)|^p d\mu(x) < +\infty$, $1\le p<\infty$.

[1]Although we were not able to prove global convergence to an optimum, the algorithm has converged to an optimum in all numerical examples we have tried.

II. Problem Formulation and Main Result

We adopt the basic framework depicted in Fig. 1, which comprises two decision makers labelled $\mathcal{U}_1$ and $\mathcal{U}_2$ and a remote estimator labelled $\mathcal{E}$ connected by a collision channel χ. There are two independent, continuous random variables X_1 and X_2 with distributions μ_1 and μ_2, which are accessible to $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. We assume that $\mathbf{E}[X_i]=0$ and $\mathbf{V}(X_i)<+\infty$, $i\in\{1,2\}$. Each decision maker has the authority to decide whether to attempt a transmission of its measurement to the estimator. It is also important to notice from Fig. 1 that there is no communication between $\mathcal{U}_1$ and $\mathcal{U}_2$, which precludes policies that involve coordination.

The collision channel defined below conveys information from the sensors to the estimators.

Definition 1 (Collision Channel): The channel input alphabet is $\mathbb{S} \stackrel{\text{def}}{=} \mathbb{R}\cup\{\varnothing\}$, and the channel output alphabet is $\mathbb{Y} \stackrel{\text{def}}{=} (\{1,2\}\times\mathbb{R})\cup\{\varnothing,\mathfrak{C}\}$, where $\mathfrak{C}$ represents the occurrence of a *collision*. The symbol $\varnothing$ indicates absence of transmission. The collision channel is a deterministic two-input map $\chi:\mathbb{S}\times\mathbb{S}\to\mathbb{Y}$ defined as follows:

$$\chi(s_1,s_2) \stackrel{\text{def}}{=} \begin{cases} (1,s_1) & \text{if } s_1\neq\varnothing, s_2=\varnothing \\ (2,s_2) & \text{if } s_1=\varnothing, s_2\neq\varnothing \\ \mathfrak{C} & \text{if } s_1\neq\varnothing, s_2\neq\varnothing \\ \varnothing & \text{if } s_1=\varnothing, s_2=\varnothing. \end{cases} \quad (1)$$

The channel inputs are denoted by S_1 and S_2, while Y designates the output that is defined as $Y \stackrel{\text{def}}{=} \chi(S_1,S_2)$.

Assumption 1: We assume that a transmission is successful if it conveys its real-valued measurement to the estimator. This is a realistic premise when the transmitted message has enough bits to represent a real number with negligible quantization error. According to (1), we also consider that each packet contains in its header the identification number of the sender. This allows the remote estimator to unambiguously determine the origin of a successful transmission.

The following are precise definitions of the communication policies at the decision makers.

Definition 2 (Communication Policies): A communication policy for the ith sensor is determined by a map $\mathcal{U}_i:\mathbb{R}\to[0,1]$. The actions U_1 and U_2 are binary random variables that satisfy the following probabilistic law:

$$\mathbf{P}(U_i=1|X_i=x_i,X_j=x_j) \stackrel{\text{def}}{=} \mathcal{U}_i(x_i), \quad i\neq j$$

where $i,j\in\{1,2\}$, for which $U_i=1$ means that the ith sensor will attempt transmission; and $U_i=0$ means that it will remain silent. We adopt independent randomization mechanisms to generate U_1 and U_2, which guarantee that (X_1, U_1) is independent of (X_2, U_2). Each U_i acts on the ith channel input according to the following definition:

$$S_i \overset{\text{def}}{=} \begin{cases} X_i & \text{if } U_i = 1 \\ \varnothing & \text{if } U_i = 0 \end{cases}, \quad i \in \{1, 2\}. \tag{2}$$

The combined action of (2) and the channel in (1) leads to the following rule to determine the output of the channel:

$$Y = \begin{cases} (1, X_1) & \text{if } U_1 = 1, U_2 = 0 \\ (2, X_2) & \text{if } U_1 = 0, U_2 = 1 \\ \mathfrak{C} & \text{if } U_1 = 1, U_2 = 1 \\ \varnothing & \text{if } U_1 = 0, U_2 = 0. \end{cases}$$

We can now precisely state the problem of optimal remote estimation over the collision channel.

Problem 1: Let X_1 and X_2 be two independent continuous random variables with zero mean and finite variance. Consider the following cost:

$$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) \overset{\text{def}}{=} \mathbf{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right] \tag{3}$$

where each $\hat{X}_i$ is defined below[2]

$$\hat{X}_i \overset{\text{def}}{=} \mathcal{E}_i(Y), \quad \text{and} \quad \mathcal{E}_i(y) \overset{\text{def}}{=} \mathbf{E}[X_i | Y = y], \quad y \in \mathbb{Y}. \tag{4}$$

Solve the following optimization problem:

$$\underset{\mathcal{U}_1 \in \mathbb{U}_1, \mathcal{U}_2 \in \mathbb{U}_2}{\text{minimize}} \; \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2)$$

where the minimization is carried over choices in $\mathbb{U}_i$, which represents the set of randomized policies defined as follows:

$$\mathbb{U}_i \overset{\text{def}}{=} \left\{\mathcal{U} \in L^2_{\mu_i}(\mathbb{R}) | \mathcal{U} : \mathbb{R} \to [0, 1]\right\}.$$

Remark 1: If X_1 and X_2 are nonconstant then the two terms in (3) cannot be both zero because the collision channel can convey at most one sensor transmission to the estimator. This implies that there is a tradeoff in (3) that causes the minimal cost to be always positive.

A. Person-by-Person Optimality

Problem 1 can be categorized as a team decision problem with continuous observations and action spaces, where decision makers DM_1 and DM_2 have access to local information and must choose their actions to jointly minimize a common cost [3]. Moreover, the information pattern is nonclassical, which in general leads to intractable nonconvex optimization problems [35]. Related problems with discrete observation and action sets are known to be NP-hard [36]. However, it may be possible to establish structural properties of jointly optimal policies (also known as *team-optimal*) by using the so-called *person-by-person* optimality approach, which is rooted on the following concept:

[2]Notice that the estimator in (4) is always optimal for the cost in (3).

Definition 3 (Person-by-Person Optimal Solutions): A pair of policies $(\mathcal{U}_1^\star, \mathcal{U}_2^\star)$ is person-by-person optimal if

$$\mathcal{J}\left(\mathcal{U}_1^\star, \mathcal{U}_2^\star\right) \le \mathcal{J}\left(\mathcal{U}_1, \mathcal{U}_2^\star\right), \;\; \mathcal{U}_1 \in \mathbb{U}_1$$
$$\mathcal{J}\left(\mathcal{U}_1^\star, \mathcal{U}_2^\star\right) \le \mathcal{J}\left(\mathcal{U}_1^\star, \mathcal{U}_2\right), \;\; \mathcal{U}_2 \in \mathbb{U}_2.$$

Using the fact that if the pair $(\mathcal{U}_1^*, \mathcal{U}_2^*)$ is jointly optimal for Problem 1 then it is also person-by-person optimal [3], if a particular structure holds for every person-by-person optimal policy, it must also hold for team-optimal policies. The advantage of using this approach is to decompose a complicated problem into simpler subproblems for which a systematic analysis is possible.

B. Optimality of Threshold Policies for Problem 1

The goal of this subsection is to state our main result as Theorem 1, which guarantees the optimality of deterministic threshold policies as defined below. This is an important structural result because it shows that the infinite dimensional minimization stated in Problem 1 can be recast as a finite-dimensional search with respect to threshold limits. We will defer the proof of Theorem 1 until the end of Section III, in which we develop all the required auxiliary results.

Definition 4 (Determinisic Threshold Policy): A policy $\mathcal{U}$ is of the deterministic threshold type if there are constants a and b in $\bar{\mathbb{R}}$ for which the following holds:

$$\mathcal{U}(x) = \begin{cases} 0 & \text{if } a \le x \le b \\ 1 & \text{otherwise} \end{cases}, \quad x \in \mathbb{R}.$$

If $a = -b$ then the threshold policy is called *symmetric*, otherwise it is called *asymmetric*.

Theorem 1: There is a pair of deterministic threshold policies $(\breve{\mathcal{U}}_1^*, \breve{\mathcal{U}}_2^*)$ that is optimal for Problem 1.

Remark 2: Although we were unable to do so, we believe that the existence of an optimum could have been established using the results in [37]. Regardless of whether this interesting connection to [37] is possible or not, Theorem 1 is an indispensable result because it shows the existence of an optimal solution with a specific deterministic threshold structure.

III. Analogy With a Problem of Remote Estimation With Communication Costs

Here, we establish an analogy between the minimization of the cost in (3) with respect to either $\mathcal{U}_1$ or $\mathcal{U}_2$ while keeping the other fixed, and a problem of optimal remote estimation systems with communication costs. The intuition behind this correspondence is that if $\mathcal{U}_{\{j:j\neq i\}}$ is fixed then the increase in the mean squared estimation error of X_j that results from the collisions caused by $\mathcal{U}_i$ can be viewed, from the perspective of $\mathcal{U}_i$, as a communication cost. This analogy will be useful in the proof of our main result (Theorem 1) and, as we explain later, it also leads to a new class of problems of independent interest.

In order to make this analogy precise, without loss of generality, consider that $\mathcal{U}_j^*$ is fixed to an arbitrary choice in $\mathbb{U}_j$, and let $\mathcal{J}_{\mathcal{U}_j^*}(\mathcal{U}_i)$ denote the resulting cost defined as follows:

$$\mathcal{J}_{\mathcal{U}_j^*}(\mathcal{U}_i) \stackrel{\text{def}}{=} \mathcal{J}(\mathcal{U}_1,\mathcal{U}_2)\Big|_{\mathcal{U}_j=\mathcal{U}_j^*}, \quad i \neq j. \tag{5}$$

The following proposition unveils the underlying additive communication cost embedded in (5).

Proposition 1 ([34]): Given a preselected $\mathcal{U}_j^*$, and $i \neq j$, the following holds:

$$\mathcal{J}_{\mathcal{U}_j^*}(\mathcal{U}_i) = \mathbf{E}\left[(X_i - \hat{X}_i)^2\right] + \rho_{\mathcal{U}_j^*}\mathbf{P}(U_i = 1) + \theta_{\mathcal{U}_j^*} \tag{6}$$

where $\rho_{\mathcal{U}_i^*}$ and $\theta_{\mathcal{U}_j^*}$ do not depend on $\mathcal{U}_i$ and are given by

$$\rho_{\mathcal{U}_j^*} = \mathbf{E}\left[(X_j - \hat{X}_j)^2|U_i = 1\right] - \mathbf{E}\left[(X_j - \hat{X}_j)^2|U_i = 0\right]$$
$$\theta_{\mathcal{U}_j^*} = \mathbf{E}\left[(X_j - \hat{X}_j)^2|U_i = 0\right].$$

The cost functional in (6) has three components: the first is the mean square estimation error of X_i, the second ascribes a cost $\rho_{\mathcal{U}_j^*}$ to the probability of attempting a transmission, and the third is constant with respect to $\mathcal{U}_i$.

The following proposition will be important later on.

Proposition 2 ([34]): For a given a preselected $\mathcal{U}_j^*$, it holds that $\rho_{\mathcal{U}_j^*} \geq 0$.

From Propositions 1 and 2, we conclude that the minimization of $\mathcal{J}$ with respect to $\mathcal{U}_i$, while keeping $\mathcal{U}_j^*$ fixed, can be cast as follows:

Problem 2: Consider that β in [0, 1] and a nonnegative constant ϱ are given. Let D and X be two independent random variables. The variable D is Bernoulli with $\mathbf{P}(D = 1) = \beta$ and X is a continuous random variable with distribution μ, zero-mean and finite variance σ_X^2. Let $\mathbb{U} \stackrel{\text{def}}{=} \{\mathcal{U} \in L^2_\mu(\mathbb{R}) | \mathcal{U} : \mathbb{R} \rightarrow [0, 1]\}$. Find a solution to the following:

$$\underset{\mathscr{U} \in \mathbb{U}}{\text{minimize}} \; \mathscr{J}(\mathscr{U}) \tag{7}$$

where the cost is defined for any $\mathscr{U}$ in $\mathbb{U}$ as follows:

$$\mathscr{J}(\mathscr{U}) \stackrel{\text{def}}{=} \mathbf{E}\left[(X - \hat{X})^2\right] + \varrho\mathbf{P}(U = 1).$$

Here, $U \in \{0, 1\}$, $\mathbf{P}(U = 1|X = x) \stackrel{\text{def}}{=} \mathscr{U}(x)$, the pair (X, U) is independent of D and $\hat{X} \stackrel{\text{def}}{=} \mathbf{E}[X|Z]$, where Z is the output of the point-to-point collision channel defined as follows:

$$Z \stackrel{\text{def}}{=} \begin{cases} X & \text{if } U = 1, D = 0 \\ \mathfrak{C} & \text{if } U = 1, D = 1 \\ \varnothing & \text{if } U = 0. \end{cases} \tag{8}$$

Remark 3: From Propositions 1 and 2, we conclude that the minimization of (3) with respect to $\mathcal{U}_i$, while keeping a preselected $\mathcal{U}_j^*$ fixed, is equivalent to Problem 2 provided that we recognize a correspondence between (X, U, D) and (X_i, U_i, U_j). To complete this analogy, we can select β as $\mathbf{P}(U_j = 1)$ and ϱ as $\rho_{\mathcal{U}_j^*}$.

A. Optimality of Deterministic Threshold Policies for Problem 2

Notice that the channel specified in (8), which is adopted in the formulation of Problem 2, is fundamentally different from the erasure model of [4]. Unlike the latter where erasures occur independently of the channel input, information loss in (8) results from collision events that depend on both U and the exogenous variable D. Our goal in what follows is to prove Theorem 2, which establishes that there is at least one deterministic threshold policy that is optimal. This is an important result for the solution of Problem 2 because it shows that the infinite dimensional optimization in (7) can be recast as a finite-dimensional minimization with respect to the thresholds. Equally important is Lemma 2, which is central for the proof of Theorem 1.

We start by stating the following proposition that can be derived from standard continuity arguments:

Proposition 3: Consider a policy $\mathscr{U}' \in \mathbb{U}$ for Problem 2. Given a positive real constant ϵ, there is a policy $\mathscr{U} \in \mathbb{U}$ satisfying the following two inequalities:

$$|\mathscr{J}(\mathscr{U}) - \mathscr{J}(\mathscr{U}')| < \epsilon \tag{9a}$$
$$0 < \mathscr{U}(x) < 1, \quad \mu - \text{a.e.} \tag{9b}$$

The following lemma will be used on the proof of Lemma 2 and it states the solution of a moment minimization problem akin to what can be found in [38].

Lemma 1: Consider that a random variable X with distribution μ, constants $\gamma \in \mathbb{R}$ and $\alpha \in (0, 1)$ are given, and the optimization problem

$$\underset{\mathscr{G} \in L^2_\mu(\mathbb{R})}{\text{minimize}} \quad \mathbf{E}\left[X^2\mathscr{G}(X)\right] \tag{10a}$$
$$\text{subject to} \quad \mathbf{E}\left[X\mathscr{G}(X)\right] = \gamma \tag{10b}$$
$$\mathbf{E}\left[\mathscr{G}(X)\right] = 1 \tag{10c}$$
$$0 \leq \mathscr{G}(x) \leq \frac{1}{1 - \alpha}, \quad \mu - \text{a.e.} \tag{10d}$$

If the problem in (10) has a feasible solution $\mathscr{G}$ for which (10d) is satisfied with strict inequalities then there is an optimal solution $\breve{\mathscr{G}}$ with the following threshold structure:

$$\breve{\mathscr{G}}(x) = \begin{cases} \frac{1}{1-\alpha} & \text{if } \breve{a} \leq x \leq \breve{b} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

for some real constants $\breve{a}$ and $\breve{b}$.

Proof: The proof uses a technique from [39, Section 5.7.3] adapted to infinite dimensional linear programming. We start by defining a new objective function $\mathscr{C} : L^2_\mu(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ that incorporates the inequality constraints by making them implicit

$$\mathscr{C}(\mathscr{G}) = \begin{cases} \mathbf{E}\left[X^2\mathscr{G}(X)\right] & \text{if } 0 \leq \mathscr{G}(x) \leq \frac{1}{1-\alpha}, \; \mu - \text{a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

This leads to the following equivalent optimization problem:

$$\begin{aligned} \inf_{\mathcal{G}\in L^2_\mu(\mathbb{R})} \quad & \mathcal{C}(\mathcal{G}) \\ \text{subject to} \quad & \mathbf{E}\left[X\mathcal{G}(X)\right] = \gamma \\ & \mathbf{E}\left[\mathcal{G}(X)\right] = 1. \end{aligned} \tag{12}$$

Letting $\nu \in \mathbb{R}^2$ denote the vector of dual variables $\nu = (\nu_0, \nu_1)$, the Lagrange dual function for this problem is

$$\mathcal{C}^*(\nu) = -\nu_1 - \nu_0\gamma + \inf_{0\le\mathcal{G}(x)\le\frac{1}{1-\alpha}} \mathbf{E}\left[(X^2+\nu_0X+\nu_1)\mathcal{G}(X)\right] \tag{14}$$

where the bounds on $\mathcal{G}$ hold μ-a.e. The following function minimizes the last term in the right hand side of (13):

$$\mathcal{G}_\nu(x) = \begin{cases} \frac{1}{1-\alpha} & \text{if } x^2+\nu_0x+\nu_1 \le 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

which when substituted in (13) leads to the following expression for $\mathcal{C}^*(\nu)$:

$$\mathcal{C}^*(\nu) = -\nu_1 - \nu_0\gamma - \frac{1}{1-\alpha}\mathbf{E}\left[\left[X^2+\nu_0X+\nu_1\right]^-\right].$$

In Appendix B it is shown in detail that, provided there exists a feasible solution $\mathcal{G}$ for which (10d) is satisfied with strict inequalities, then strong duality holds and there exists a vector $\nu^* \in \mathbb{R}^2$ that maximizes $\mathcal{C}^*(\nu)$. Hence, an optimal solution $\mathcal{G}^*$ for the problem in (12) is obtained by substituting such a ν^* in (14), or equivalently, by setting the equality $\mathcal{G}^* = \mathcal{G}_{\nu^*}$. From Appendix C, any such ν^* leads to a polynomial $x^2 + \nu_0^*x + \nu_1^*$ that always admits real roots, which we denote as a^* and b^*, with $a^* \le b^*$. Since $x^2 + \nu_0^*x + \nu_1^*$ is a convex parabola in x, we can further conclude that the test $x^2 + \nu_0x + \nu_1 \le 0$ can be replaced with $a^* \le x \le b^*$. Using these facts in conjunction with (14), we conclude that there is an optimal solution of the form in (11). ■

Lemma 2: Assume that $\mathcal{U}' \in \mathbb{U}$ is a given policy for Problem 2. For every positive real constant ϵ, there is a deterministic threshold policy $\breve{\mathcal{U}}$ for which $\mathcal{J}(\breve{\mathcal{U}}) < \mathcal{J}(\mathcal{U}') + \epsilon$.

Proof: Our overarching strategy is to view the problem in (10) as a version of Problem 2 with additional constraints so that we can use Lemma 1 to obtain the desired result.

From Proposition 3, we know that from the given $\mathcal{U}'$, we can construct $\mathcal{U}''$ so that the following holds:

$$\mathcal{J}(\mathcal{U}'') < \mathcal{J}(\mathcal{U}') + \epsilon \tag{15a}$$

$$0 < \mathcal{U}''(x) < 1, \quad \mu - \text{a.e.} \tag{15b}$$

We now proceed by defining γ and α as follows:

$$\gamma \stackrel{\text{def}}{=} \mathbf{E}[X|U''=0] \tag{16a}$$

$$\alpha \stackrel{\text{def}}{=} \mathbf{P}(U''=1) = \mathbf{E}\left[\mathcal{U}''(X)\right] \tag{16b}$$

where the action U'' is generated from the policy $\mathcal{U}''$ as described in Problem 2. The cases when $\alpha = 0$ or $\alpha = 1$ immediately correspond to optimal threshold policies $\breve{\mathcal{U}}(x) = 0$, μ-a.e. and $\breve{\mathcal{U}}(x) = 1$, μ-a.e., respectively. So, without loss of generality, we consider that α is in (0, 1).

Fact 1: Given γ and α defined in (16), we conclude that $\mathcal{G}''$ defined as follows:

$$\mathcal{G}''(x) \stackrel{\text{def}}{=} \frac{1-\mathcal{U}''(x)}{1-\alpha}, \quad x \in \mathbb{R}$$

satisfies the constraints of (10), and from (15b) it also satisfies (10d) strictly. Hence, we conclude that the conditions for Lemma 1 are satisfied for the γ and α defined in (16). Denote with $\mathbb{U}_{\alpha,\gamma}$ the subset of policies $\mathcal{U} \in \mathbb{U}$ for which the following holds:

$$\mathbf{E}[X|U=0] = \gamma; \quad \mathbf{E}\left[\mathcal{U}(X)\right] = \alpha \tag{17}$$

where U is the action generated from $\mathcal{U}$ as described in Problem 2. For any $\mathcal{U}$ in $\mathbb{U}_{\alpha,\gamma}$, $\mathcal{J}(\mathcal{U})$ can be written as

$$\mathcal{J}(\mathcal{U}) = \mathbf{E}\left[\beta(X-\hat{x}_{\mathfrak{C}})^2 + \varrho|U=1\right]\alpha + \mathbf{E}\left[(X-\hat{x}_\varnothing)^2|U=0\right](1-\alpha)$$

for which $\hat{x}_{\mathfrak{C}}$ and $\hat{x}_\varnothing$ are defined as

$$\hat{x}_{\mathfrak{C}} \stackrel{\text{def}}{=} \mathbf{E}[X|Z=\mathfrak{C}]; \quad \hat{x}_\varnothing \stackrel{\text{def}}{=} \mathbf{E}[X|Z=\varnothing]$$

where Z is the channel output as described in Problem 2. Since $\mathbf{E}[X|U=1] = \hat{x}_{\mathfrak{C}}$, we can rewrite the cost as

$$\mathcal{J}(\mathcal{U}) = (1-\beta)\mathbf{E}[X^2|U=0](1-\alpha) - \left[\frac{(1-\alpha)^2}{\alpha}\beta + (1-\alpha)\right]\gamma^2 + \varrho\alpha + \beta\sigma_X^2 \tag{18}$$

where $\mathcal{U} \in \mathbb{U}_{\alpha,\gamma}$ and we used the facts that $\hat{x}_{\mathfrak{C}}\mathbf{P}(U=1) = -\hat{x}_\varnothing\mathbf{P}(U=0)$ and $\hat{x}_\varnothing = \mathbf{E}[X|U=0] = \gamma$.

Fact 2: Notice that for $\mathcal{U}$ in $\mathbb{U}_{\alpha,\gamma}$, $\mathbf{E}[X^2|U=0]$ and $\mathbf{E}[X|U=0]$ can be written as $\mathbf{E}[X^2\mathcal{G}(X)]$ and $\mathbf{E}[X\mathcal{G}(X)]$, respectively, where $\mathcal{G}$ is found from Bayes' law to be

$$\mathcal{G}(x) = \frac{1-\mathcal{U}(x)}{1-\alpha}, \quad x \in \mathbb{R}. \tag{19}$$

Fact 3: From Fact 2, (19) and Section III-A, we conclude that minimizing $\mathcal{J}(\mathcal{U})$ with respect to $\mathcal{U}$ constrained to (17) is equivalent to solving the problem in (10).

From Fact 1 we know that the conditions for the validity of Lemma 1 are satisfied. Hence, from Lemma 1, Fact 3 and (19) we conclude that there is a deterministic threshold policy $\breve{\mathcal{U}}$ that minimizes $\mathcal{J}(\mathcal{U})$ subject to the constraints in (17). Such policy can be computed from the solution in Lemma 1 as follows:

$$\breve{\mathcal{U}}(x) = 1 + (\alpha-1)\breve{\mathcal{G}}(x), \quad x \in \mathbb{R}.$$

Since $\breve{\mathcal{U}}$ satisfies (17), by optimality we conclude that $\mathcal{J}(\breve{\mathcal{U}}) \le \mathcal{J}(\mathcal{U}'')$ holds, which in conjunction with (15a) concludes the proof. ■

Theorem 2: There is a deterministic threshold policy $\breve{\mathcal{U}}^*$ that is optimal for Problem 2.

Proof: Consider that the parameters that specify an instance of Problem 2 are given and denote the minimum or infimum in (7) as ς^*. Let $\mathcal{U}_{(n)}$ be a sequence of policies such that $\lim_{n\to\infty} \mathscr{J}(\mathcal{U}_{(n)}) = \varsigma^*$. From Lemma 2, we can define a sequence of threshold policies $\breve{\mathcal{U}}_{(n)}$, such that

$$\mathscr{J}\left(\breve{\mathcal{U}}_{(n)}\right) \leq \mathscr{J}\left(\mathcal{U}_{(n)}\right) + \frac{1}{n+1}. \tag{20}$$

Let $\breve{a}_{(n)}$ and $\breve{b}_{(n)}$ be the thresholds associated with $\breve{\mathcal{U}}_{(n)}(x)$. We now proceed to study the convergence to an optimum based on the sequence $\{(\breve{a}_{(n)}, \breve{b}_{(n)})\}_{n=0}^{\infty}$. We start by remarking that the sequence $\{(\breve{a}_{(n)}, \breve{b}_{(n)})\}_{n=0}^{\infty}$ has at least one subsequence $\{(\breve{a}_{(m_n)}, \breve{b}_{(m_n)})\}_{n=0}^{\infty}$ for which $\breve{a}^* \stackrel{\text{def}}{=} \lim_{n\to\infty} \breve{a}_{(m_n)}$ and $\breve{b}^* \stackrel{\text{def}}{=} \lim_{n\to\infty} \breve{b}_{(m_n)}$ are well defined and take values in $\bar{\mathbb{R}}$, with $\breve{a}^* \leq \breve{b}^*$. The proof follows by using (20) and Proposition 7 (Appendix A) to conclude that the thresholds $\breve{a}^*$ and $\breve{b}^*$ define an optimal policy for Problem 2, which we denote as $\breve{\mathcal{U}}^*$. ■

B. Proof of Theorem 1

Our proof is organized in two main steps that hinge on the analogy developed in the first part of this section, which presents results of independent interest for an optimal point-to-point remote estimation paradigm that includes communication costs.

Proof of Theorem 1: For any parameter selection that specify an instance of Problem 1, let ς^* be the infimum or minimum cost and select a sequence of policies $\{(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)})\}_{n=0}^{\infty}$ for which the following holds:

$$\lim_{n\to\infty} \mathcal{J}\left(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)}\right) = \varsigma^*. \tag{21}$$

Step 1: From Remark 3 and Lemma 2, we conclude that there is a sequence of deterministic threshold policies $\{(\breve{\mathcal{U}}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$ for which the following holds:

$$\mathcal{J}\left(\breve{\mathcal{U}}_{1,(n)}, \mathcal{U}_{2,(n)}\right) \leq \mathcal{J}\left(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)}\right) + \frac{1}{n+1}, \quad n \geq 0 \tag{22a}$$

$$\mathcal{J}\left(\mathcal{U}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)}\right) \leq \mathcal{J}\left(\mathcal{U}_{1,(n)}, \mathcal{U}_{2,(n)}\right) + \frac{1}{n+1}, \quad n \geq 0. \tag{22b}$$

Step 2: We can repeat the method used in Step 1 to conclude that there is a sequence of deterministic threshold policies $\{(\breve{\breve{\mathcal{U}}}_{1,(n)}, \breve{\breve{\mathcal{U}}}_{2,(n)})\}_{n=0}^{\infty}$ for which the following holds:

$$\mathcal{J}\left(\breve{\mathcal{U}}_{1,(n)}, \breve{\breve{\mathcal{U}}}_{2,(n)}\right) \leq \mathcal{J}\left(\breve{\mathcal{U}}_{1,(n)}, \mathcal{U}_{2,(n)}\right) + \frac{1}{n+1}, \quad n \geq 0 \tag{23a}$$

$$\mathcal{J}\left(\breve{\breve{\mathcal{U}}}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)}\right) \leq \mathcal{J}\left(\mathcal{U}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)}\right) + \frac{1}{n+1}, \quad n \geq 0. \tag{23b}$$

Our conclusion from (21) to (23) is that the sequences $\{(\breve{\mathcal{U}}_{1,(n)}, \breve{\breve{\mathcal{U}}}_{2,(n)})\}_{n=0}^{\infty}$ and $\{(\breve{\breve{\mathcal{U}}}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$ satisfy the following:

$$\lim_{n\to\infty} \mathcal{J}\left(\breve{\breve{\mathcal{U}}}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)}\right) = \varsigma^* \tag{24a}$$

$$\lim_{n\to\infty} \mathcal{J}\left(\breve{\mathcal{U}}_{1,(n)}, \breve{\breve{\mathcal{U}}}_{2,(n)}\right) = \varsigma^*. \tag{24b}$$

Without loss of generality, we proceed to analyze the convergence of $\{(\breve{\breve{\mathcal{U}}}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$ to an optimal solution. An equivalent argument could have been developed using $\{(\breve{\mathcal{U}}_{1,(n)}, \breve{\breve{\mathcal{U}}}_{2,(n)})\}_{n=0}^{\infty}$. Let $\breve{\breve{a}}_1^*$, $\breve{\breve{b}}_1^*$, $\breve{a}_2^*$ and $\breve{b}_2^*$ be constants in $\bar{\mathbb{R}}$ for which there is a subsequence $\{(\breve{\breve{\mathcal{U}}}_{1,(m_n)}, \breve{\mathcal{U}}_{2,(m_n)})\}_{n=0}^{\infty}$ whose associated thresholds satisfy $\lim_{n\to\infty} \breve{\breve{a}}_{1,(m_n)} = \breve{\breve{a}}_1^*$, $\lim_{n\to\infty} \breve{\breve{b}}_{1,(m_n)} = \breve{\breve{b}}_1^*$, $\lim_{n\to\infty} \breve{a}_{2,(m_n)} = \breve{a}_2^*$ and $\lim_{n\to\infty} \breve{b}_{2,(m_n)} = \breve{b}_2^*$. The proof is concluded by invoking Proposition 6 (Appendix A) to show that the thresholds $\breve{\breve{a}}_1^*$, $\breve{\breve{b}}_1^*$, $\breve{a}_2^*$ and $\breve{b}_2^*$ define an optimal policy, which we denote as $(\breve{\breve{\mathcal{U}}}_1^*, \breve{\mathcal{U}}_2^*)$. ■

Remark 4: Note that the proofs of the structural results of Theorems 1 and 2 are completely independent of the distributions of X_1 and X_2, as long as they are zero mean independent continuous random variables with finite variances. In fact, X_1 and X_2 may come from completely different families of distributions. The structural result of Theorem 1 is also true for a sensing system with any number of sensors measuring mutually independent random variables, under the additional assumption that the remote estimator can decode the indices of the sensors involved in a collision.

IV. Policy Design Via Quantization Theory

On the second part of the paper we turn our focus to the design of optimal policies for Problem 2, which will ultimately lead to person-by-person optimal policies for Problem 1. We base our arguments on the observation that Problem 2 can be understood as an one-bit quantization problem with a nonuniform distortion metric across the two quantization regions. The intuition behind this interpretation comes from the fact that the sensor's decision of transmitting or not can be exploited for communication by encoding in the two possible actions an additional bit of information. When the transmission is successful, this additional bit is redundant because the received packet already contains all the relevant information for a perfect estimate. However, when the transmission fails due to the occurrence of a collision, the estimator estimates X from this additional bit. The objective of the system's designer is to "compress" in this bit (represented by $\varnothing$ and $\mathfrak{C}$) the maximum amount of information about X as possible.

Remark 5: This situation does not occur if instead of collisions we had random erasures. The observation of an erasure does not reveal the sensor's intent to communicate since they cannot be distinguished from "erasures" due to the absence of transmitted packets when the channel is idle.

A. One-Bit Quantization

The structural result of Theorem 2 established the existence of an optimal deterministic threshold policy for Problem 2. Here we will make the analogy with one-bit quantization more precise. First, we will let $\mathcal{U}$ act as a deterministic encoder, which partitions the real line $\mathbb{R}$ into two measurable sets $\mathbb{A}_0$ and $\mathbb{A}_1 = \mathbb{A}_0^c$ such that $\mathbb{A}_0 \overset{\text{def}}{=} \mathcal{U}^{-1}(0)$ and $\mathbb{A}_1 \overset{\text{def}}{=} \mathcal{U}^{-1}(1)$. We relax Problem 2 by letting the estimator $\mathcal{E}$ lie in a class of admissible deterministic decoders, where

$$\mathcal{E}(z) = z, \quad \textbf{if} \quad z \notin \{\varnothing, \mathfrak{C}\}.$$

Let $\hat{x}_\varnothing, \hat{x}_\mathfrak{C} \in \mathbb{R}$, define

$$\mathcal{E}(\varnothing) \overset{\text{def}}{=} \hat{x}_\varnothing \quad \text{and} \quad \mathcal{E}(\mathfrak{C}) \overset{\text{def}}{=} \hat{x}_\mathfrak{C}.$$

With the cost in Problem 2 now depending on $\mathcal{U}$ and $\mathfrak{E}$, and assuming that the random variable X has zero mean, finite variance σ_X^2, and admits a probability density function $f_X(x)$, we can rewrite the functional as

$$\tilde{\mathcal{J}}(\mathcal{U}, \mathcal{E}) = \int_{\mathbb{A}_0} (x - \hat{x}_\varnothing)^2 f_X(x)dx + \int_{\mathbb{A}_0^c} \left[\beta(x - \hat{x}_\mathfrak{C})^2 + \varrho\right] f_X(x)dx.$$

Our goal is to choose a partition of $\mathbb{R}$ into a measurable set $\mathbb{A}_0$ and its complement $\mathbb{A}_0^c$ and their respective representation points $\hat{x}_\varnothing$ and $\hat{x}_\mathfrak{C}$ such as to minimize the average distortion quantified by $\tilde{\mathcal{J}}(\mathcal{U}, \mathcal{E})$.

Applying the structural result obtained in the previous section the optimal partition is such that $\mathbb{A}_0 = [a, b]$, where $a \leq b$ and $a, b \in \bar{\mathbb{R}}$. When they exist, the optimal thresholds and representation symbols can be found by solving the optimization problem in

$$\begin{aligned} \underset{\hat{x}_\varnothing, \hat{x}_\mathfrak{C}, a, b \in \mathbb{R}}{\text{minimize}} \quad & \int_{[a,b]} (x - \hat{x}_\varnothing)^2 f_X(x)dx + \int_{[a,b]^c} \left[\beta(x - \hat{x}_\mathfrak{C})^2 + \varrho\right] f_X(x)dx \\ \text{subject to} \quad & a \leq b \end{aligned} \tag{25}$$

with variables in $a, b, \hat{x}_\varnothing, \hat{x}_\mathfrak{C} \in \mathbb{R}$. In other words, the communication and estimation policies that jointly minimize the cost $\tilde{\mathcal{J}}(\mathcal{U}, \mathcal{E})$ can be found by solving an optimal scalar quantization problem of a random variable $X \sim f_X(x)$, where representation symbols are penalized by distinct quadratic distortion functions.

Remark 6: Relaxing the estimator to lie in a larger class of admissible estimators, rather than fixing it as the conditional expectation operator, will be important in order to obtain a numerical procedure for finding solutions for Problem 2.

B. The Nearest Neighbor Condition and an Equivalent Problem

Let $\hat{x} \overset{\text{def}}{=} (\hat{x}_\varnothing, \hat{x}_\mathfrak{C}) \in \mathbb{R}^2$. For any given $\hat{x}$, the set $\mathbb{A}_0^*$ which yields the minimal cost must satisfy

$$x \in \mathbb{A}_0^* \Leftrightarrow (x - \hat{x}_\varnothing)^2 \leq \beta(x - \hat{x}_\mathfrak{C})^2 + \varrho.$$

This is true regardless of the probability density function $f_X(x)$. Since $\varrho \geq 0$, for $0 \leq \beta < 1$, the second degree polynomial

$$\mathcal{P}_{\hat{x}}(x) \overset{\text{def}}{=} (x - \hat{x}_\varnothing)^2 - \beta(x - \hat{x}_\mathfrak{C})^2 - \varrho$$

admits two distinct real roots.[3] We will denote the minimum of these roots by $a(\hat{x})$ and the largest by $b(\hat{x})$. Therefore, without loss of optimality, we may assume that the no-transmission interval is, for a given $\hat{x}$

$$\mathbb{A}_0 = [a(\hat{x}), b(\hat{x})]$$

and the cost is reduced to a function $\mathcal{J}_q : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\mathcal{J}_q(\hat{x}) \overset{\text{def}}{=} \int_{[a(\hat{x}), b(\hat{x})]} (x - \hat{x}_\varnothing)^2 f_X(x)dx + \int_{[a(\hat{x}), b(\hat{x})]^c} \left[\beta(x - \hat{x}_\mathfrak{C})^2 + \varrho\right] f_X(x)dx \tag{26}$$

where the two maps $a : \mathbb{R}^2 \to \mathbb{R}$ and $b : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$a(\hat{x}) \overset{\text{def}}{=} \frac{1}{1-\beta}\left[(\hat{x}_\varnothing - \beta\hat{x}_\mathfrak{C}) - \sqrt{\beta(\hat{x}_\varnothing - \hat{x}_\mathfrak{C})^2 + (1-\beta)\varrho}\right] \tag{27}$$

$$b(\hat{x}) \overset{\text{def}}{=} \frac{1}{1-\beta}\left[(\hat{x}_\varnothing - \beta\hat{x}_\mathfrak{C}) + \sqrt{\beta(\hat{x}_\varnothing - \hat{x}_\mathfrak{C})^2 + (1-\beta)\varrho}\right]. \tag{28}$$

Remark 7: The function $\mathcal{J}_q(\hat{x})$ is twice continuously differentiable at every $\hat{x} \in \mathbb{R}^2$. Furthermore, there is no loss in optimality in minimizing $\mathcal{J}_q(\hat{x})$ over $\mathbb{R}^2$ instead of solving the problem in (25), which is defined over $\mathbb{R}^4$ [40].

Therefore, we have an equivalent finite dimensional unconstrained optimization problem in terms of the pair of representation points $\hat{x}$ that specify the estimator $\mathcal{E}$:

Problem 3: Given the constants $\varrho \geq 0$, $\beta \in [0, 1)$ and $f_X(x)$, solve the unconstrained nonlinear optimization problem

$$\text{minimize} \quad \mathcal{J}_q(\hat{x})$$

with variable $\hat{x} \in \mathbb{R}^2$.

C. The Centroid Condition

We now obtain a set of necessary optimality conditions corresponding to $\nabla \mathcal{J}_q(\hat{x}^*) = 0$.

Proposition 4: Any minimizing $\hat{x}^* = (\hat{x}_\varnothing^*, \hat{x}_\mathfrak{C}^*)$ must satisfy

$$\int_{[a(\hat{x}^*), b(\hat{x}^*)]} \left(x - \hat{x}_\varnothing^*\right) f_X(x)dx = 0$$

$$\int_{[a(\hat{x}^*), b(\hat{x}^*)]^c} (x - \hat{x}_\mathfrak{C}^*) f_X(x)dx = 0.$$

[3] When $\beta = 1$, the polynomial $\mathcal{P}_{\hat{x}}(x)$ admits a single root. This case can be arbitrarily well approximated by a sequence of problems with $\beta_n \in [0, 1)$ such that $\{\beta_n\} \uparrow 1$.

Proposition 4 essentially states that the optimal representation points must be centroids of the interval $\mathbb{A}_0^* = [a(\hat{x}^*), b(\hat{x}^*)]$, defined by the two roots of $\mathcal{P}_{\hat{x}^*}(x)$, and its complement. If the density f_X has full support on $\mathbb{R}$, the conditions in Proposition 4 can be written more compactly as

$$\hat{x}^* = \mathcal{F}(\hat{x}^*)$$

where $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\mathcal{F}(\hat{x}) \overset{\text{def}}{=} \begin{bmatrix} \frac{1}{\int_{a(\hat{x})}^{b(\hat{x})} f_X(x)dx} \\ \frac{-1}{1-\int_{a(\hat{x})}^{b(\hat{x})} f_X(x)dx} \end{bmatrix} \int_{a(\hat{x})}^{b(\hat{x})} x f_X(x)dx. \tag{29}$$

Hence, any critical point of $\mathcal{J}_q(\hat{x})$ and, in particular, any optimal solution $\hat{x}^*$ are fixed-points of the nonlinear map $\mathcal{F}$.

Proposition 5: If f_X has full support on $\mathbb{R}$ and is even, the following statements about the map $\mathcal{F}$ in (29) hold:

(i) any nonzero fixed point $\hat{x}$ must satisfy

$$\text{sgn}(\hat{x}_\varnothing) = -\text{sgn}(\hat{x}_\mathfrak{C})$$

(ii) the vector $\hat{x} = (0, 0)$ is always a fixed point;

(iii) if $\hat{x}$ is a fixed point then $-\hat{x}$ is also a fixed point;

(iv) the set

$$\mathbb{L}_\beta \overset{\text{def}}{=} \{\hat{x} \in \mathbb{R}^2 | \hat{x}_\varnothing = \beta \hat{x}_\mathfrak{C}\} \tag{30}$$

is mapped into (0,0);

(v) any fixed point $\hat{x}$ satisfies

$$|\hat{x}_\varnothing||\hat{x}_\mathfrak{C}| \le \sigma_X^2$$

where $\sigma_X^2 = \mathbf{V}(X)$.

Proof: The proof can be found in [41]. ■

It is easy to show that if f_X is an even probability density function, then the cost $\mathcal{J}_q(\hat{x})$ is an even function. In particular, if $X \sim \mathcal{N}(0, \sigma_X^2)$ the cost function in (26) is even. One important consequence this fact together with Proposition 5 is that, for any even density f_X, the search for an optimal solution $\hat{x}^*$ may be constrained to either

$$\mathbb{Q}_1 \overset{\text{def}}{=} \{\hat{x} \in \mathbb{R}^2 | \hat{x}_\varnothing \ge 0, \hat{x}_\mathfrak{C} \le 0\}$$

or

$$\mathbb{Q}_2 \overset{\text{def}}{=} \{\hat{x} \in \mathbb{R}^2 | \hat{x}_\varnothing \le 0, \hat{x}_\mathfrak{C} \ge 0\}$$

without loss of optimality. Fig. 2 shows where the stationary points of $\mathcal{F}$ may lie.

V. Numerical Examples and the Optimality of Asymmetric Threshold Policies

In this section, we provide examples of optimal policies for Problem 2 obtained as solutions to Problem 3 when $X \sim \mathcal{N}(0, \sigma_X^2)$.

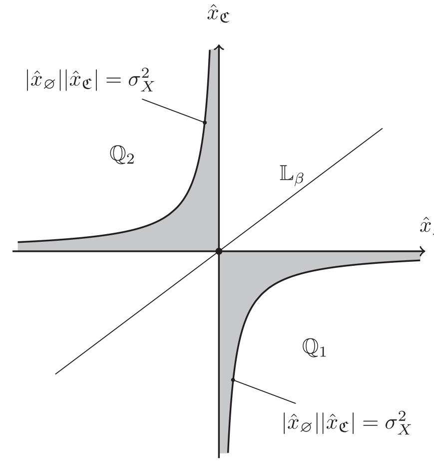

Fig. 2. The shaded region of $\mathbb{R}^2$ shown above contains all the critical points of $\mathcal{J}_q(\hat{x})$ when f_X is an even density. The origin is always a critical point and the line $\mathbb{L}_\beta$ is entirely mapped by $\mathcal{F}$ into (0, 0).

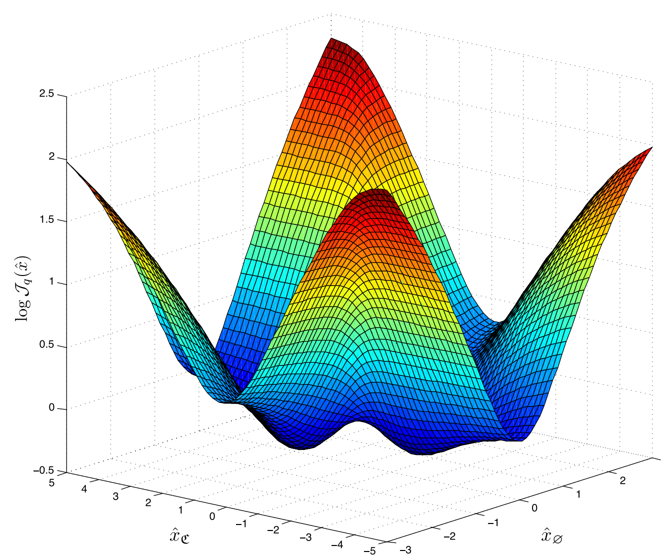

Fig. 3. Cost function $\mathcal{J}_q(\hat{x})$ in log-scale for $\sigma_X^2 = 1, \beta = 0.5$, and $\varrho = 1$. The log-scale helps us in visualizing the two minima.

Example 1 (Nonconvexity of the Cost Function): Consider the cost $\mathcal{J}_q(\hat{x})$ for a typical choice of parameters: let $X \sim \mathcal{N}(0, 1)$, $\beta = 0.5$ and $\varrho = 1$ in Problem 3. The plot of the cost function in log-scale is shown in Fig. 3 and its level curves are shown in Fig. 4. These two figures allow us to make two important observations. First, since the sublevel sets are not convex, $\mathcal{J}_q(\hat{x})$ is neither convex nor quasi-convex. This is the case even if we constrain its domain to $\mathbb{Q}_1$ or $\mathbb{Q}_2$. The second observation is the occurrence of a single minimum in each $\mathbb{Q}_i$, $i \in \{1, 2\}$. However, due to the intricate structure of $\mathcal{J}_q(\hat{x})$, obtaining a proof of this fact remains an open problem.

The optimal solutions to the various minimization problems considered in what follows were obtained using standard nonlinear programming solvers constraining $\mathcal{J}_q(\hat{x})$ to $\mathbb{Q}_1$. More sophisticated algorithms for solving Problem 3 with optimality guarantees (such as the Branch-and-bound method) can be used along with the fact that $\mathcal{J}_q(\hat{x})$ can be decomposed as a difference of convex functions since it is twice continuously

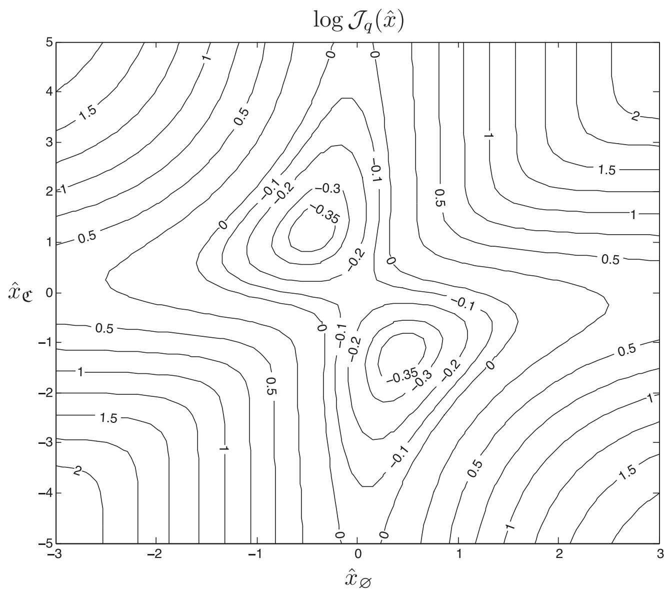

Fig. 4. Level curves of cost function $\mathcal{J}_q(\hat{x})$ in log-scale for $\sigma_X^2 = 1$, $\beta = 0.5$ and $\varrho = 1$. From the sublevel sets, we can conclude that $\mathcal{J}_q(\hat{x})$ is neither convex nor quasi-convex. Despite this fact, we can observe that there is a single minimum in $\mathbb{Q}_1$ and another in $\mathbb{Q}_2$.

differentiable [42]. However, in view of the observations in Example 1, this is beyond the scope of this paper.

Example 2 (Optimality of Asymmetric Thresholds): Let $X \sim \mathcal{N}(0,1)$, $\beta = 0.5$ and $\varrho = 1$ in Problem 3. A pair of representation points that minimize the cost function in (26) is $\hat{x}^* = (0.434, -1.255)$ corresponding to a cost $\mathcal{J}_q^* = 0.681$. By using the expressions in (27) and (28), we obtain the values of the optimal thresholds of the optimal no-transmission interval $\mathbb{A}_0^* = [-0.653, 4.898]$. Therefore, the optimal no-transmission interval is characterized by asymmetric thresholds. If, on the other hand, we consider only symmetric threshold policies, by the centroid conditions, their optimal representation points are $(0,0) \overset{\text{def}}{=} \hat{x}_{\text{sym}}$. Hence, the optimal cost within the class of symmetric policies is

$$\mathcal{J}_{\text{sym}}^* \overset{\text{def}}{=} \mathcal{J}_q(0,0) = 0.871 > \mathcal{J}_q^*.$$

We cast the observation about the asymmetry of the optimal thresholds drawn from Example 2 as the following remark.

Remark 8: For $\beta > 0$ the optimal communication policies for Problem 2 have, in general, asymmetric thresholds.

This may lead us to erroneously assume that when $\beta > 0$ the optimal policies must be asymmetric. The purpose of the next example is to show this is not always the case.

Example 3 (Optimality of Symmetric Thresholds): Consider Problem 2 with $X \sim \mathcal{N}(0,1)$, $\beta = 0.1$ and $\varrho = 1$. The pair of representation points that minimize the cost function in (26) is $\hat{x}^* = (0,0)$, yielding a cost $\mathcal{J}_q^* = 0.595$. Recovering the corresponding optimal no-transmission interval we have $\mathbb{A}_0^* = [-1.054, 1.054]$.

It is also interesting to observe how the optimal thresholds behave with the variance σ_X^2. Table I shows that when σ_X^2 increases, the no-transmission interval has a positive drift and its length increases while its probability decreases. This means that transmissions will occur more often as the variance of the random variable X increases.

TABLE I
NUMERICAL RESULTS OF THE OPTIMIZATION PROBLEM 2 WHEN $X \sim \mathcal{N}(0, \sigma_X^2)$, $\beta = 0.1$ AND $\varrho = 1$ FOR DIFFERENT VALUES OF σ_X^2

σ_X^2	$\hat{x}^*$	$\mathbb{A}_0^*$	$\mathcal{J}_q^*$	$\mathbf{P}(\mathbb{A}_0^*)$
1	$(0,0)$	$[-1.054, 1.054]$	0.595	0.708
2	$(0.505, -0.565)$	$[-0.495, 1.743]$	0.822	0.528
3	$(1.124, -0.789)$	$[0.086, 2.586]$	0.970	0.412
4	$(1.658, -0.856)$	$[0.562, 3.313]$	1.085	0.340
5	$(2.130, -0.886)$	$[0.970, 3.960]$	1.186	0.294

VI. MODIFIED LLOYD–MAX ALGORITHM

In the last part of the paper we propose an iterative procedure inspired by the Lloyd–Max algorithm [43] to design optimal communication and estimation policies for Problem 2. We call this procedure the *modified Lloyd–Max* (MLM) algorithm. The MLM is an alternative to standard nonlinear solvers to find optimal solutions to Problem 3. The kth iteration of the MLM algorithm consists of two steps.

- **Threshold update step**: For a fixed pair of representation points $\hat{x}^{(k)} \in \mathbb{R}^2$, update the thresholds that define the no-transmission interval according to

$$\mathbb{A}_0^{(k)} = \left[a\left(\hat{x}^{(k)}\right), b\left(\hat{x}^{(k)}\right)\right].$$

- **Centroid computation step**: Obtain a new pair of representation points $\hat{x}^{(k+1)}$ as the centroids of $\mathbb{A}_0^{(k)}$ and its complement, i.e.,

$$\hat{x}_{\varnothing}^{(k+1)} = \mathbf{E}\left[X | X \in \mathbb{A}_0^{(k)}\right]$$

$$\hat{x}_{\mathfrak{C}}^{(k+1)} = \mathbf{E}\left[X | X \notin \mathbb{A}_0^{(k)}\right].$$

Henceforth, we will consider only the Gaussian case by assuming that $X \sim \mathcal{N}(0, \sigma_X^2)$. This allows us to make important claims and observations about the MLM algorithm, properties of its fixed points and its convergence.

A. An Equivalent Nonlinear Autonomous Dynamical System

The MLM algorithm outlined above can be understood as a nonlinear dynamical system described by successive applications of the map $\mathcal{F}$ in (29). For a fixed $\hat{x}^{(0)} \neq (0,0)$

$$\hat{x}^{(k+1)} = \mathcal{F}\left(\hat{x}^{(k)}\right), \quad k = 0, 1, \ldots \tag{31}$$

It is important that the initial point $\hat{x}^{(0)}$ is a nonzero vector, otherwise the algorithm outputs a sequence identically equal to zero. When $X \sim \mathcal{N}(0, \sigma_X^2)$, it can be shown that the sets $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are invariant to the map $\mathcal{F}$, i.e.,

$$\mathcal{F}(\mathbb{Q}_i) \subset \mathbb{Q}_i, \quad i \in \{1, 2\}.$$

Therefore, a sequence of points generated by (31) will either belong to $\mathbb{Q}_1$ or $\mathbb{Q}_2$ depending on the initial condition $\hat{x}^{(0)}$. Furthermore, it is a well-known fact that the Lloyd–Max iterations generate a nonincreasing sequence of values of the objective function [44], i.e.,

$$\mathcal{J}_q\left(\hat{x}^{(k+1)}\right) \leq \mathcal{J}_q\left(\hat{x}^{(k)}\right), \quad k = 0, 1, \ldots$$

B. On the Convergence of the MLM Algorithm

In general, unless it is known that $\mathcal{F}$ is a Banach contraction, there are no guarantees that the dynamical system describing the MLM algorithm will converge to a unique fixed point. Moreover, empirical evidence shows that for a very large set of parameters, there are multiple fixed points. Therefore, it is unlikely that such contraction properties will hold for $\mathcal{F}$. However, the fact that the MLM is a descent algorithm together with the fact that the stationary points in $\mathbb{Q}_i$, $i \in \{1, 2\}$ are isolated indicate that convergence results to a local minimum may be proved. The paper by Du *et al.* [45] present several convergence results for Lloyd–Max type algorithms that can be used to establish convergence of the MLM. In particular, [45, Theorem 2.6] states:

> *If the iterations in the Lloyd algorithm stay in a compact set where the Lloyd map $\mathcal{F}$ is continuous, then the algorithm is globally convergent to a critical point of $\mathcal{J}_q(\hat{x})$.*

It is possible to show the existence of such a compact set when $X \sim \mathcal{N}(0, \sigma_X^2)$, $\beta \in [0, 1)$ and $\varrho > 0$. Under these conditions, the MLM is globally convergent to a local minimum of $\mathcal{J}_q(\hat{x})$ [41].

Remark 9: The classic sufficient condition due to Fleischer in [40] stating that if f_X is a log-concave density with full support on $\mathbb{R}$, the Lloyd–Max algorithm converges to a unique stationary point does not hold here due to the nonuniformity of the distortion metric in the quantization problem in (25).

C. On the Complexity of the MLM Algorithm

The MLM algorithm is executed *offline* to optimize the thresholds and representation points that specify the optimal transmission and estimation policies proposed here. Each iteration of the algorithm corresponds to the map (29) being evaluated. First, the thresholds $a(\hat{x})$ and $b(\hat{x})$ are computed, requiring a total of 5 multiplications and 5 additions. Then, 2 integrals are numerically evaluated.[4] Finally, 1 addition and 2 divisions complete the iteration. Therefore, each iteration of the MLM algorithm requires a total of: 6 additions, 5 multiplications, 2 divisions, and 2 integrations. Furthermore, it has been shown that the Lloyd–Max algorithm for any number of quantization levels runs in polynomial time (see [46] and references therein). Since our algorithm has the same structure of the traditional Lloyd–Max, the same property also holds for the MLM, imposing no barriers for its implementation.

D. Numerical Results

When $X \sim \mathcal{N}(0, \sigma_X^2)$ the design of the transmission thresholds and representation points can be done by means of a globally convergent algorithm, which consists of iteratively applying a nonlinear map $\mathcal{F}$ to a nonzero initial vector $\hat{x}^{(0)} \in \mathbb{Q}_i$, $i \in \{1, 2\}$. Since $\mathcal{J}_q(\hat{x})$ is nonconvex we are not able to claim that a critical point found through the MLM algorithm is a global minimum, but from observing the general shape of $\mathcal{J}_q(\hat{x})$ for several combination of parameters, we conjecture that $\mathcal{F}$ will have at most two critical points in each $\mathbb{Q}_i$, $i = 1, 2$. One of the stationary points is always (0, 0), which can be a global minimum in some cases. For example, the trajectory of a sequence $\{\hat{x}^{(k)}\} \to (0, 0)$ generated by the MLM applied to $\hat{x}^{(0)} = (1, -1)$ when $\beta = 0.1$ and $\varrho = 1$ is shown in Fig. 5. The stopping criterion used was based on the magnitude of the gradient at $\hat{x}^{(k)}$ as follows:

$$\left\| \nabla \mathcal{J}_q\left(\hat{x}^{(k)}\right) \right\| < 10^{-6}. \tag{32}$$

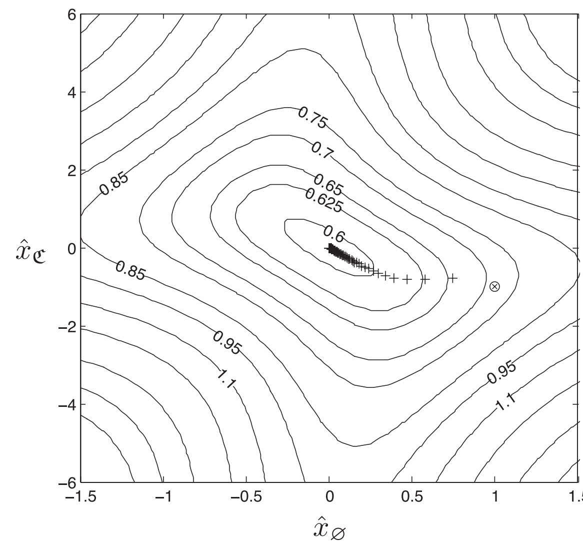

Fig. 5. Trajectory of the sequence generated by the MLM algorithm when $\beta = 0.1$, $\varrho = 1$, $\sigma_X^2 = 1$. The level curves indicate that $\mathcal{J}_q(\hat{x})$ has a single critical point at (0, 0). The initial condition $\hat{x}^{(0)}$ is shown in $\otimes$ and the remaining points are displayed using $+$.

In most cases, however, the global minimum is a nonzero stationary point, which will correspond to asymmetric thresholds for the no-transmission interval $\mathbb{A}_0^*$. Fig. 6 illustrates the trajectory of points generated by $\mathcal{F}$ when $\beta = 0.3$, $\varrho = 1$ and $\sigma_X^2 = 1$. The initial condition $\hat{x}^{(0)}$ was chosen to lie on the curve $\hat{x}_{\varnothing}\hat{x}_{\mathfrak{C}} = -\sigma_X^2$. In all the numerical examples of this section, the algorithm was initialized with $\hat{x}^{(0)} = (\sigma_X, -\sigma_X)$.

We obtained the optimal solutions of Problem 3 with the probability of collision β varying from zero to 0.99, which are displayed in Table II. A few observations can be drawn from this table. First, we notice that when $\beta = 0.1$ the number of iterations N_{it} to achieve the convergence criterion in (32) is much larger than for any other row. This is justified by the values of the cost function evaluated near the origin being very close to the minimum. Therefore, all the points around (0, 0) are nearly stationary, hence the slow convergence.

Another interesting observation is that when the probability of a concurrent transmission β approaches 1, the no-transmission interval $\mathbb{A}_0^*$ tends to increase. However, we can see that its probability tends to a value bounded away from 1.

[4] In the Gaussian case, these integrals can be expressed in closed form using the complementary error (erfc) and exponential (exp) functions.

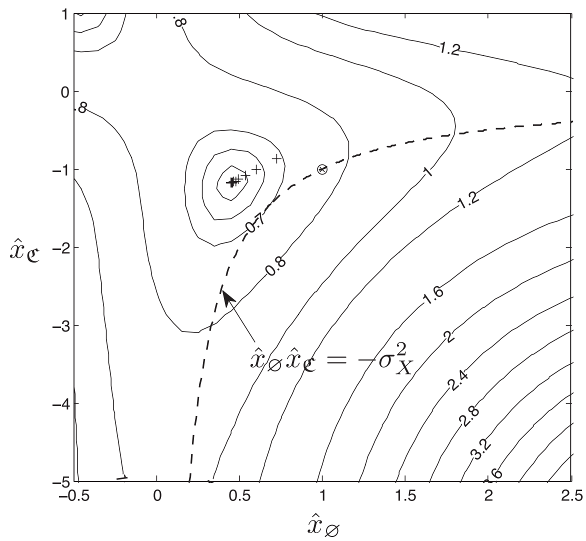

Fig. 6. Sequence of points generated by $\mathcal{F}$ converging to $\hat{x}^* = (0.454, -1.183)$. In this case, the parameters are $\beta = 0.3$, $\varrho = 1$ and $\sigma_X^2 = 1$. The initial condition, represented by $\otimes$, lies on the curve $\hat{x}_{\varnothing}\hat{x}_{\mathfrak{C}} = -\sigma_X^2$.

TABLE II
NUMERICAL RESULTS OF THE OPTIMIZATION PROBLEM 2 WHEN $\varrho = 1$, $\sigma_X^2 = 1$ AND DIFFERENT VALUES OF β

β	$\hat{x}^*$	$\mathbb{A}_0^*$	$\mathbf{P}(\mathbb{A}_0^*)$	$\mathcal{J}_q^*$	N_{it}
0	$(0, 0)$	$[-1, 1]$	0.683	0.516	39
0.1	$(0, 0)$	$[-1.054, 1.054]$	0.708	0.595	168
0.2	$(0.408, -1.008)$	$[-0.608, 2.133]$	0.712	0.649	27
0.3	$(0.454, -1.183)$	$[-0.596, 2.908]$	0.723	0.666	21
0.4	$(0.447, -1.231)$	$[-0.624, 3.756]$	0.734	0.675	22
0.5	$(0.434, -1.255)$	$[-0.653, 4.898]$	0.743	0.681	22
0.6	$(0.421, -1.275)$	$[-0.680, 6.612]$	0.752	0.687	22
0.7	$(0.409, -1.295)$	$[-0.705, 9.477]$	0.760	0.693	22
0.8	$(0.399, -1.313)$	$[-0.729, 15.22]$	0.767	0.699	22
0.9	$(0.388, -1.330)$	$[-0.752, 32.47]$	0.774	0.704	23
0.99	$(0.380, -1.345)$	$[-0.772, 343.1]$	0.780	0.709	23

TABLE III
NUMERICAL RESULTS FOR PROBLEM 2 WHEN $\beta = 0.5$ AND $\sigma_X^2 = 5$ FOR VARIOUS VALUES OF ϱ

ϱ	$\hat{x}^*$	$\mathbb{A}_0^*$	$\mathbf{P}(\mathbb{A}_0^*)$	$\mathcal{J}_q^*$	N_{it}
0	$(2.360, -1.279)$	$[0.853, 11.14]$	0.351	1.244	29
0.1	$(2.321, -1.309)$	$[0.800, 11.10]$	0.361	1.309	29
0.2	$(2.283, -1.339)$	$[0.744, 11.07]$	0.370	1.372	28
0.5	$(2.169, -1.432)$	$[0.580, 10.96]$	0.398	1.557	28
1	$(1.986, -1.591)$	$[0.311, 10.82]$	0.445	1.846	25
2	$(1.655, -1.917)$	$[-0.206, 10.66]$	0.537	2.355	27
5	$(0.970, -2.806)$	$[-1.460, 10.95]$	0.743	3.406	23
10	$(0.457, -3.866)$	$[-2.795, 12.35]$	0.894	4.246	16
20	$(0.135, -5.223)$	$[-4.377, 15.36]$	0.975	4.792	10
50	$(0.006, -7.686)$	$[-7.078, 22.47]$	0.999	4.993	5

Therefore, even when the collision event has probability 1, it may be worth paying the communication cost to transmit a packet because the optimal strategy always conveys one-bit of information through the collision and no-transmission symbols.

The dependency of the optimal solutions with the communication cost ϱ when $\beta = 0.5$ and $\sigma_X^2 = 5$ is shown in Table III. We make the following observations. Even when communication is free ($\varrho = 0$), the optimal no-transmission interval has a positive probability. This is because there is a probability that information will be lost due to a collision. In order to make the best use of the virtual signaling channel (see Fig. 7), the Shannon entropy $H(U)$ must be nonzero, which forces $\mathbf{P}(U = 0) = \mathbf{P}(\mathbb{A}_0^*) > 0$.

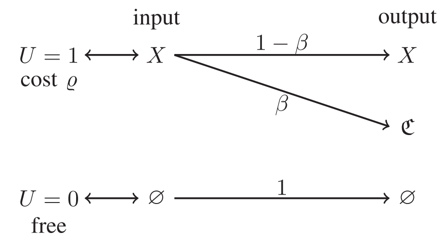

Fig. 7. Implicit channel between the sensor and the remote estimator in Problem 2.

TABLE IV
PERSON-BY-PERSON OPTIMAL POLICIES FOR DM_1 AND DM_2 IN PROBLEM 1 WHERE THE MEASUREMENTS ARE INDEPENDENTLY DISTRIBUTED AS $X_1 \sim \mathcal{N}(0, \sigma_1^2)$ AND $X_2 \sim \mathcal{N}(0, 1)$

σ_1^2	$\mathbb{A}_{1,0}^*$	$\mathbb{A}_{2,0}^*$	$\mathbf{P}(\mathfrak{C})$	$\mathbf{P}(\varnothing)$	$\mathcal{J}^*$
1	$[0.098, 5.359]$	$[0.098, 5.359]$	0.290	0.213	0.540
2	$[0.864, 4.534]$	$[-0.545, 10.50]$	0.214	0.191	0.764
3	$[1.877, 4.060]$	$[-1.260, 27.67]$	0.090	0.116	0.889
4	$[2.635, 4.158]$	$[-1.718, 56.71]$	0.040	0.072	0.945
5	$[3.236, 4.374]$	$[-2.051, 98.63]$	0.019	0.048	0.971

When $\varrho \to +\infty$ two notable things happen. One is that the number of iterations required to achieve the convergence criterion in (32) decreases sharply.

Also, as opposed to the case when $\beta \to 1$, the probability of the no-transmission interval tends to one, $\mathbf{P}(\mathbb{A}_0^*) \to 1$, as ϱ increases. Therefore, not transmitting will turn out to be optimal in the regime of very large communication costs.

E. Person-by-Person Optimal Policies for Problem 1

Table IV illustrates that the results developed in the previous sections can be used to obtain person-by-person optimal policies for Problem 1 when $X_i \sim \mathcal{N}(0, \sigma_i^2)$, $i \in \{1, 2\}$. Letting $\sigma_2^2 = 1$ and varying σ_1^2, we applied the MLM algorithm, alternating between the optimization of the policies for DM_1 and DM_2 until a fixed point was found. We do not have a proof that this procedure converges globally, but the policies obtained can be verified to be person-by-person optimal. The communication policy for DM_i is summarized by its no-transmission interval denoted by $\mathbb{A}_{i,0}$.

We observe that as the variance of the observations of DM_1 increases, the person-by-person optimal policies are such that the channel will be more often accessed by DM_1 and less often accessed by DM_2. This will cause a decrease in the probabilities of collisions and of an idle channel. It is interesting to note that all the policies listed in Table IV outperform traditional sensor scheduling policies in which only the sensor measuring the random variable with the largest variance, which is the one that can reduce the cost the most, transmits. In that case, even with collisions, the person-by-person optimal policies when $\sigma_1^2 = \sigma_2^2 = 1$ outperform the naive scheduling policy by approximately 46%. That is the case even when $\sigma_1^2 = 5$, which is considerably larger than $\sigma_2^2 = 1$, yielding a gain of approximately 3% over the scheduling policy. Also note that when $\sigma_1^2 = \sigma_2^2 = 1$, the framework of Problem 1 is completely symmetric, i.e., X_1 and X_2 are identically distributed and f_X

is an even probability density function. However, the person-by-person optimal policies have asymmetric thresholds. This is a major departure from the previous results in remote estimation from [29]–[33], all of which establish the optimality of symmetric threshold policies.

F. On the Optimality of Asymmetric Thresholds

An interesting feature of the person-by-person policies of Table IV is the asymmetry of optimal thresholds. This can be intuitively justified by the presence of a collision symbol at the channel output, which can be used to convey information to the remote estimator. Note that when the probability of the channel being occupied in Problem 2 is zero, the optimal policies are always symmetric. The presence of two distinct symbols for collision and no-transmission creates an implicit noiseless channel between the sensor and estimator shown in Fig. 7. For a given communication cost, asymmetric communication policies can lower the variance of the estimation error in Problem 2 and, consequently, may be optimal for Problem 1. Notice that for the models in [4], in which collisions and communication costs are not considered, it is optimal to always transmit.

VII. Conclusion

We studied the collision channel as a model for interference in a multisensor remote estimation problem. Our goal was to characterize optimal communication policies and the tradeoffs that occur in the design of optimal communication policies in a simple interference setting. We characterized the threshold structure of optimal policies using a person-by-person optimality approach. We showed that, from the perspective of a single decision maker, the aggregate quadratic cost can be decomposed in two terms: a mean squared estimation error and a communication cost. For this cost, we proved that the optimal communication policies have a threshold structure. Policy design is then related to a problem of finding an optimal one-bit quantizer for the sensor's measurement. An iterative scheme that can be used to find locally optimal thresholds was proposed.

A. Consequences for More General Classes of Systems

In most multiagent sequential decision making problems, it is useful to solve the problem for a single pair of agents in a single time step to characterize the structure of the optimal policies. This paper takes a first step in solving a decentralized sequential estimation problem over a collision channel, providing valuable insights into the nature of its optimal solutions. Despite the apparent simplicity of our model, our results have implications for a wide class of systems. First, the structural results in this paper also hold for continuous random vectors, requiring only minor modifications in the proofs. We chose to present the results for scalar random variables to simplify the proofs and facilitate the visualization of the threshold policies in $\mathbb{R}$, since in $\mathbb{R}^n$ they would become hyper-ellipsoidal surfaces. Another important feature of our formulation is that the results hold for an arbitrary number of sensor nodes measuring independent random variables. In order to see this, we need the additional assumption that the remote estimator can decode the index of the sensors that were involved in a collision. Then, when solving the person-by-person optimization problem, we can treat the sensors with fixed strategies as a "superuser" observing a random vector and occupying the channel with a given probability β. Following the same arguments of Section III we obtain the same structural result.

The one-shot problem we have solved is a fundamental building block for the sequential problem. The fact that the result is independent of the probability density functions of the measurements is particularly important in the sequential case with feedback because the state of a Gauss-Markov process conditioned on the channel outputs or acknowledgements has a distribution that is no longer Gaussian. Finally, we have shown that the optimal thresholds can be asymmetric for a single stage problem with even probability density functions. This implies that the restriction to symmetric threshold policies is suboptimal for more general sequential event-based estimation problems over the collision channel.

B. Future Work

There are many opportunities for future work that branch out from the problems studied here. One important question is to investigate whether person-by-person optimal solutions for Problem 1 are in fact team-optimal or not. A natural extension of the model presented here is to allow the channel to support N users and a collision event when M simultaneous transmissions are made. Another open question is to determine the structure of optimal policies when observations are correlated instead of independent. Sequential versions of Problem 1 and Problem 2 are interesting and challenging problems, which are related to sequential one-bit quantization schemes such as sigma-delta modulators.

Appendix A
Auxiliary Results on Continuity

This Appendix includes two propositions that state important continuity properties of the costs for Problems 1 and 2. In particular, they state that when evaluated for deterministic threshold policies, the cost varies continuously with respect to the thresholds. This is observation is key to show the existence of an optimum in Theorems 1 and 2.

Proposition 6: Let $(\bar{\mathcal{U}}_1, \bar{\mathcal{U}}_2)$ be a given pair of deterministic threshold policies characterized by thresholds $\bar{a}_1$, $\bar{b}_1$, $\bar{a}_2$ and $\bar{b}_2$ in $\bar{\mathbb{R}}$. Let $\{(\breve{\mathcal{U}}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)})\}_{n=0}^{\infty}$ be a given sequence of policies with associated thresholds $\{\breve{a}_{1,(n)}\}_{n=0}^{\infty}$, $\{\breve{b}_{1,(n)}\}_{n=0}^{\infty}$, $\{\breve{a}_{2,(n)}\}_{n=0}^{\infty}$ and $\{\breve{b}_{2,(n)}\}_{n=0}^{\infty}$. If $\lim_{n\to\infty} \breve{a}_{1,(n)} = \bar{a}_1$, $\lim_{n\to\infty} \breve{b}_{1,(n)} = \bar{b}_1$, $\lim_{n\to\infty} \breve{a}_{2,(n)} = \bar{a}_2$ and $\lim_{n\to\infty} \breve{b}_{2,(n)} = \bar{b}_2$ holds then the following also holds:

$$\lim_{n\to\infty} \mathcal{J}\left(\breve{\mathcal{U}}_{1,(n)}, \breve{\mathcal{U}}_{2,(n)}\right) = \mathcal{J}(\bar{\mathcal{U}}_1, \bar{\mathcal{U}}_2). \quad (33)$$

Proposition 7: Let $\bar{\mathcal{U}}$ be a deterministic threshold policy characterized by thresholds $\bar{a}$ and $\bar{b}$ in $\bar{\mathbb{R}}$, with $\bar{a} \leq \bar{b}$. Let

$\mathcal{U}_{(n)}$ be a sequence of policies for problem 2 with associated thresholds $a_{(n)}$ and $b_{(n)}$ that satisfy $\lim_{n\to\infty} a_{(n)} = \bar{a}$ and $\lim_{n\to\infty} b_{(n)} = \bar{b}$. The following holds:

$$\lim_{n\to\infty} \mathcal{J}\left(\mathcal{U}_{(n)}\right) = \mathcal{J}(\bar{\mathcal{U}}). \tag{34}$$

Appendix B
Strong Duality

The purpose of this Appendix is to provide a proof that, under the conditions of Lemma 1, strong duality holds for the problem in (10). This is important since, as opposed to their finite dimensional counterparts, strong duality for infinite dimensional linear programs does not necessarily hold. Our proof will hinge on a result due to Borwein and Lewis [47] adapted by Limber and Goodrich in [48]. The constraint qualification under which strong duality holds involves the concepts of quasi interior (**qi**) and quasi-relative interior (**qri**) of a set. The relative interior of a set is denoted by **ri**.

Theorem 3 (Limber and Goodrich [48]—Theorem 4.1): Let $\mathbb{G}$ be a Banach space, $\mathcal{J} : \mathbb{G} \to (-\infty, +\infty]$ a convex functional, $\mathcal{A} : \mathbb{G} \to \mathbb{R}^n$ a linear continuous map, and $\mathbb{G}_c \subset \mathbb{G}$ a closed convex set. Let $b \in \mathbb{R}^n$ be a fixed vector such that $b \in \mathbf{ri}\, \mathcal{A}(\mathbb{G}_c)$ and suppose that

$$p^* = \inf\left\{\mathcal{J}(\mathcal{g}) | \mathcal{A}(\mathcal{g}) = b,\ \mathcal{g} \in \mathbb{G}_c\right\}$$

is finite. If

$$d^* = \sup_{\nu\in\mathbb{R}^n}\left\{b^\mathsf{T}\nu + \inf_{\mathcal{g}\in\mathbb{G}_c}\left\{\mathcal{J}(\mathcal{g}) - \nu^\mathsf{T}\mathcal{A}(\mathcal{g})\right\}\right\}$$

then $p^* = d^*$, i.e., strong duality holds and the maximum is attained at some $\nu^* \in \mathbb{R}^n$. Since $\mathbf{ri}\, \mathcal{A}(\mathbb{G}_c) = \mathcal{A}(\mathbf{qri}\, \mathbb{G}_c)$, when $\mathbf{qri}\, \mathbb{G}_c \neq \emptyset$, this can be restated as follows: if

$$\exists \mathcal{g} \in \mathbf{qri}\, \mathbb{G}_c \text{such that } A(\mathcal{g}) = b$$

then $p^* = d^*$.

Proof: The reader is referred to [47] and [48]. ∎

We will verify that the conditions of Theorem 3 are indeed satisfied for the optimization problem in (10):

(i) The space $L^2_\mu(\mathbb{R})$ is a Banach space.

(ii) The objective functional is linear in $\mathcal{g}$ and therefore convex.

(iii) The map $\mathcal{A} : L^2_\mu(\mathbb{R}) \to \mathbb{R}^2$, where $\mathcal{A}(\mathcal{g}) \stackrel{\text{def}}{=} [\mathbf{E}[X\mathcal{g}(X)]\mathbf{E}[\mathcal{g}(X)]]^\mathsf{T}$ is linear in $\mathcal{g}$, and it is also bounded and therefore continuous. Boundedness can be verified as follows.:

$$\begin{aligned}\|\mathcal{A}(\mathcal{g})\|_2^2 &= \left|\mathbf{E}\left[X\mathcal{g}(X)\right]\right|^2 + \left|\mathbf{E}\left[\mathcal{g}(X)\right]\right|^2 \\ &\leq \left(\mathbf{E}[X^2] + 1\right)\mathbf{E}\left[|\mathcal{g}(X)|^2\right] < +\infty.\end{aligned}$$

The first inequality follows from the Cauchy-Schwarz inequality applied to the first term and Jensen's inequality applied to the second term. The strict inequality in the last step follows from the fact that X has finite second moment and $\mathcal{g} \in L^2_\mu(\mathbb{R})$.

(iv) The set

$$\mathbb{G}_c = \left\{\mathcal{g} \in L^2_\mu(\mathbb{R}) | 0 \leq \mathcal{g}(x) \leq \frac{1}{1-\alpha},\ \mu - \text{a.e.}\right\}$$

is closed and convex.

(v) Assuming the existence of a feasible point we have

$$p^* \leq \mathbf{E}\left[X^2\mathcal{g}(X)\right] \leq \frac{1}{1-\alpha}\mathbf{E}[X^2] < +\infty$$

where the strict inequality follows from X having finite second moment.

(vi) Finally, we must check if the following constraint qualification is satisfied:

$$\exists \mathcal{g} \in \mathbf{qri}\, \mathbb{G}_c \text{ such that } \mathcal{A}(\mathcal{g}) = b$$

which corresponds to Borwein–Lewis' constraint qualification in [48]. Therefore, in order to have strong duality, there must be feasible point $\mathcal{g} \in \mathbf{qri}\, \mathbb{G}_c$ such that $\mathcal{A}(\mathcal{g}) = [\gamma\ 1]^\mathsf{T}$. From [48, Theorem 2.1, Example 2.2], we have

$$\mathbf{qri}\, \mathbb{G}_c = \left\{\mathcal{g} \in L^2_\mu(\mathbb{R}) \middle| 0 < \mathcal{g}(x) < \frac{1}{1-\alpha},\ \mu - \text{a.e.}\right\}$$

which is the condition that must be satisfied for Lemma 1 to hold.

Appendix C

In this Appendix we state and prove the following result used in the proof of Lemma 1.

Proposition 8: If ν^* is a maximizer of the the Lagrange dual function in (13), the polynomial $x^2 + \nu_0^* x + \nu_1^*$ always admits distinct real roots.

Proof: We will show that $(\nu_0^*)^2 > 4\nu_1^*$. Suppose that ν satisfies $\nu_0^2 \leq 4\nu_1$, implying that $[x^2 + \nu_0 x + \nu_1]^- \equiv 0$. The Lagrange dual function becomes $\mathcal{C}^*(\nu) = -\nu_1 - \nu_0\hat{x}_\varnothing$, its supremum subject to $4\nu_1 \geq \nu_0^2$ is equal to $\hat{x}_\varnothing^2$ and it is achieved by $\nu_0^* = -2\hat{x}_\varnothing$ and $\nu_1^* = \hat{x}_\varnothing^2$.

When $\nu_0^2 > 4\nu_1$, the polynomial $x^2 + \nu_0 x + \nu_1$ admits two distinct real roots denoted by $a(\nu)$ and $b(\nu)$

$$a(\nu), b(\nu) = \frac{-\nu_0 \pm \sqrt{\nu_0^2 - 4\nu_1}}{2}.$$

Let $\nu_0 = -2\hat{x}_\varnothing$ and $\nu_1 = \hat{x}_\varnothing^2 - \delta$, for some $\delta > 0$. Clearly, $\nu_0^2 - 4\nu_1 = 4\delta > 0$. We will show that $\exists \delta > 0$ such that $\mathcal{C}^*(\nu) > \hat{x}_\varnothing^2$ and therefore there is no loss in optimality in restricting the dual problem to $\{\nu \in \mathbb{R}^2 | \nu_0^2 > 4\nu_1\}$. We start with

$$\mathcal{C}^*(\nu)\Bigg|_{\substack{\nu_0=-2\hat{x}_\varnothing \\ \nu_1=\hat{x}_\varnothing^2-\delta}} = \hat{x}_\varnothing^2 + \delta + \frac{1}{1-\alpha}\int_{\hat{x}_\varnothing-\sqrt{\delta}}^{\hat{x}_\varnothing+\sqrt{\delta}} \left[(x-\hat{x}_\varnothing)^2 - \delta\right] f_X(x)dx$$

and define

$$\int_{\hat{x}_\varnothing-\sqrt{\delta}}^{\hat{x}_\varnothing+\sqrt{\delta}} (x-\hat{x}_\varnothing)^2 f_X(x)dx \stackrel{\text{def}}{=} \mathcal{W}(\delta).$$

When x varies from $\hat{x}_\varnothing - \sqrt{\delta}$ to $\hat{x}_\varnothing + \sqrt{\delta}$, the quantity $(x - \hat{x}_\varnothing)^2$ varies from 0 to δ. Therefore

$$0 \leq \mathcal{W}(\delta) \leq \int_{\hat{x}_\varnothing - \sqrt{\delta}}^{\hat{x}_\varnothing + \sqrt{\delta}} \delta f_X(x) dx \stackrel{\text{def}}{=} \mathcal{V}(\delta).$$

Since

$$\frac{\mathcal{V}(\delta)}{\delta} = \int_{\hat{x}_\varnothing - \sqrt{\delta}}^{\hat{x}_\varnothing + \sqrt{\delta}} f_X(x) dx$$

the limit $\delta \downarrow 0$ yields $\mathcal{V}(\delta)/\delta \to 0$. Therefore, $\mathcal{V}(\delta) = o(\delta)$ and consequently, $\mathcal{W}(\delta) = o(\delta)$. Implying that

$$\mathcal{C}^*(\nu)\Big|_{\substack{\nu_0 = -2\hat{x}_\varnothing \\ \nu_1 = \hat{x}_\varnothing^2 - \delta}} = \hat{x}_\varnothing^2 + \delta + o(\delta) > \hat{x}_\varnothing^2.$$

■

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Marcos M. Vasconcelos received the B.Sc. and M.Sc. degrees in electrical engineering from the Federal University of Pernambuco (UFPE), Recife, Brazil, in 2004 and 2006, respectively. He is currently working toward the Ph.D. degree in electrical and computer engineering at the University of Maryland, College Park, MD, USA.

His current research interests include networked control and estimation, optimization, and information theory.

Mr. Vasconcelos is a recipient of a CAPES/Fulbright scholarship.

Nuno C. Martins received the M.S. degree in electrical engineering from I.S.T., Portugal, in 1997, and the Ph.D. degree in electrical engineering and computer science with a minor in mathematics from the Massachusetts Institute of Technology (MIT), Cambridge, MA, USA, in 2004. He also concluded a Financial Technology Option program at Sloan School of Management (MIT), in 2004.

He is currently an Associate Professor of Electrical and Computer Engineering at the University of Maryland at College Park, College Park, MA, USA, where he also holds a joint appointment with the Institute for Systems Research. He was Director of the Maryland Robotics Center from 2012 until 2014. His research interests include distributed control, team decision, optimization, networked control, and communications.

Dr. Martins received a National Science Foundation CAREER Award in 2007, the 2006 American Automatic Control Council O. Hugo Schuck Award, the 2010 Outstanding ISR Faculty Award, and the 2008 IEEE CSS Axelby Award for an outstanding paper in the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He was a Member of the Editorial Board of *Systems and Control Letters* (Elsevier), *Automatica*, and of the IEEE Control Systems Society Conference Editorial Board. He was also a program Vice-Chair for the IEEE Conference on Decision and Control in 2013 and 2014.