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Collision Channel

Capacity Region and a Mutual Information Game

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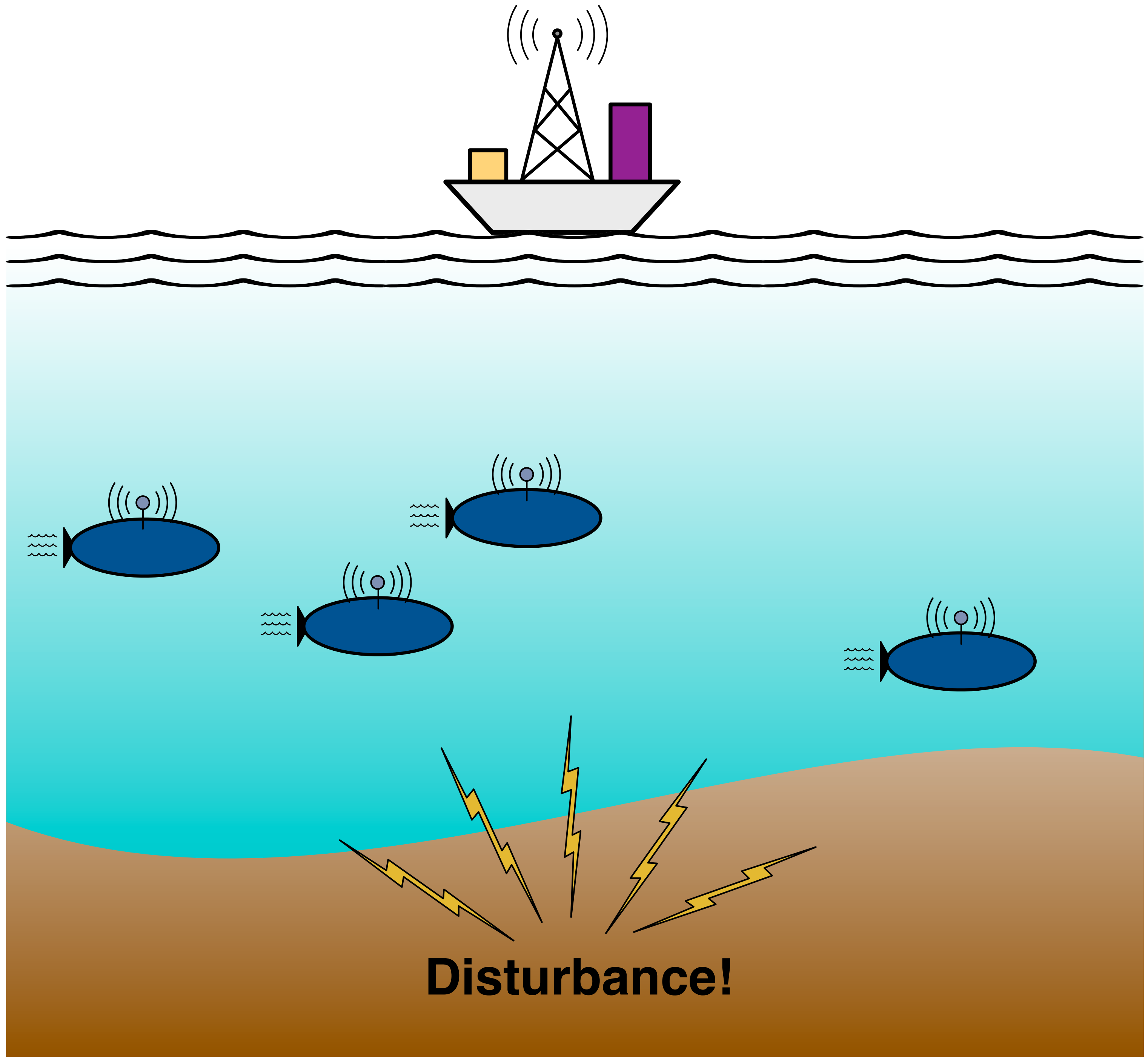
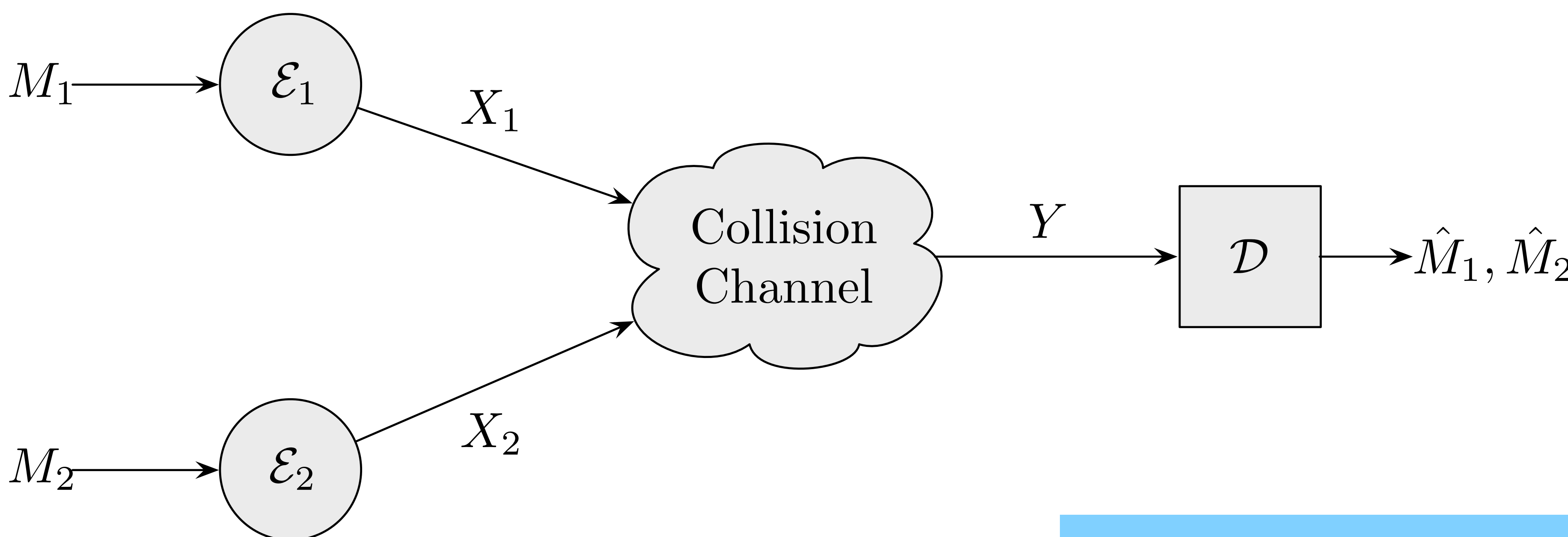
Underwater acoustic sensor networks



Packet collisions **Long propagation delays** ⟶ **Lack of feedback**

Collision channel

- **Widely used in wireless communications**
- >1 simultaneous transmission results in a **collision**
- Users decide whether to transmit or not



Input alphabet

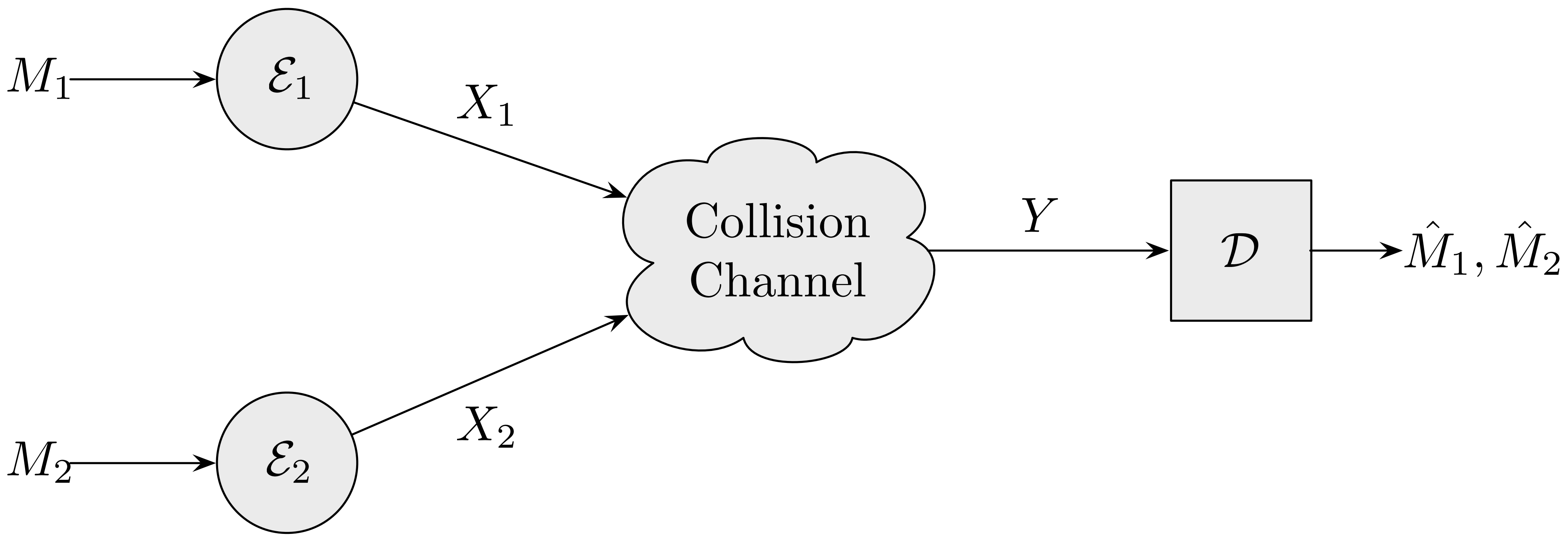
$$\mathcal{X}_i = \{\mathbf{S}, x_i(1), x_i(2), \cdots, x_i(q_i)\}, \quad i \in \{1, 2\}$$

$q_i + 1$ symbols

$\mathbf{S}$ is the **silent symbol**

q_i **non-silent** symbols
($\log q_i$ **payload** bits)

Collision channel



Silent symbol: $\mathbf{S}$

Collision symbol: $\mathbf{C}$

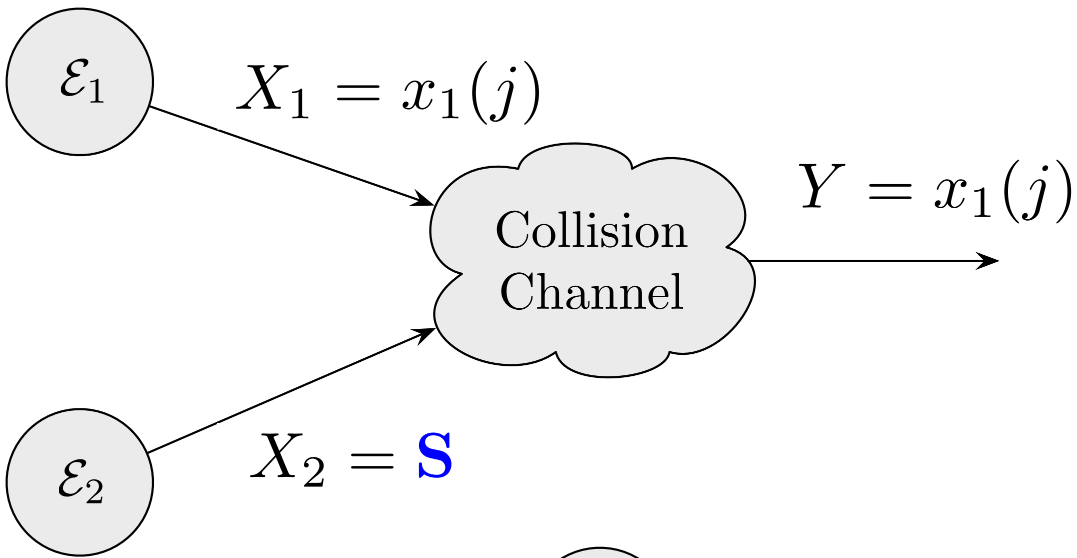
$$\mathcal{X}_i = \{\mathbf{S}, x_i(1), x_i(2), \cdots, x_i(q_i)\}, \quad i \in \{1, 2\}$$

$$\mathcal{Y} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \{\mathbf{C}\}$$

$$\boxed{\mathcal{X}_1 \cap \mathcal{X}_2 = \{\mathbf{S}\}}$$

Packets from different users have different headers!

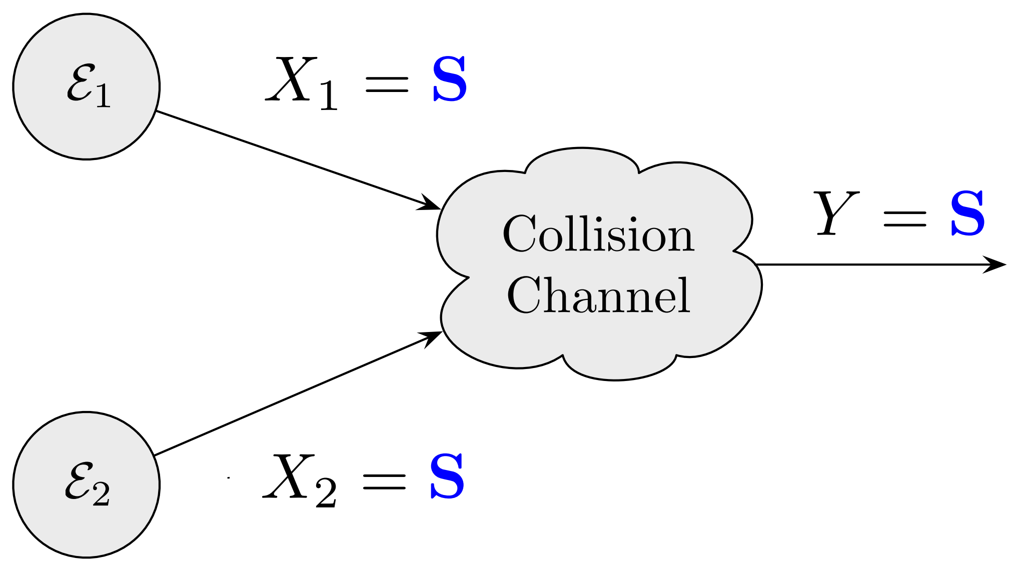
Collision channel



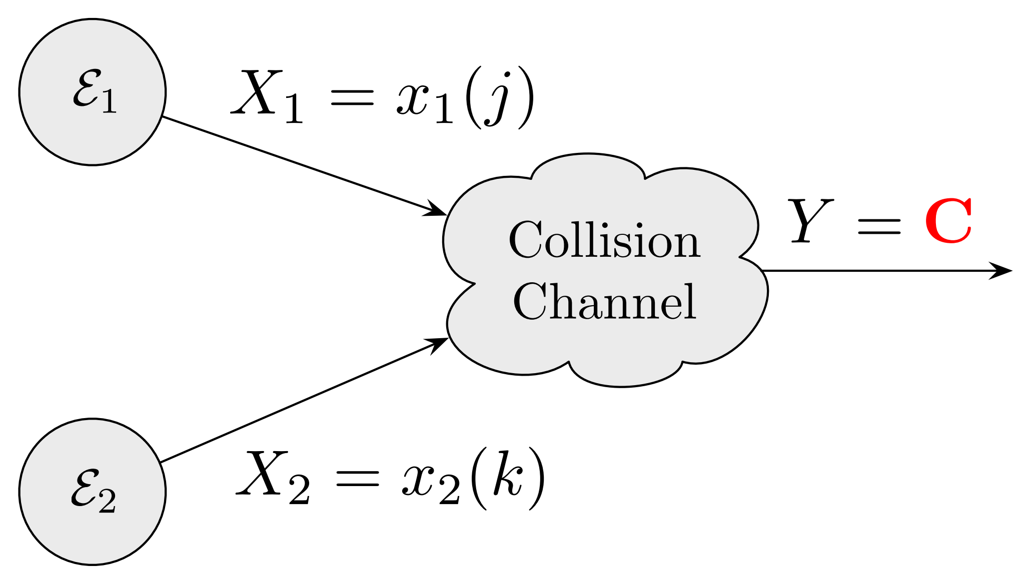
Success

Deterministic channel

$$H(Y|X_1, X_2) = 0$$



Silence



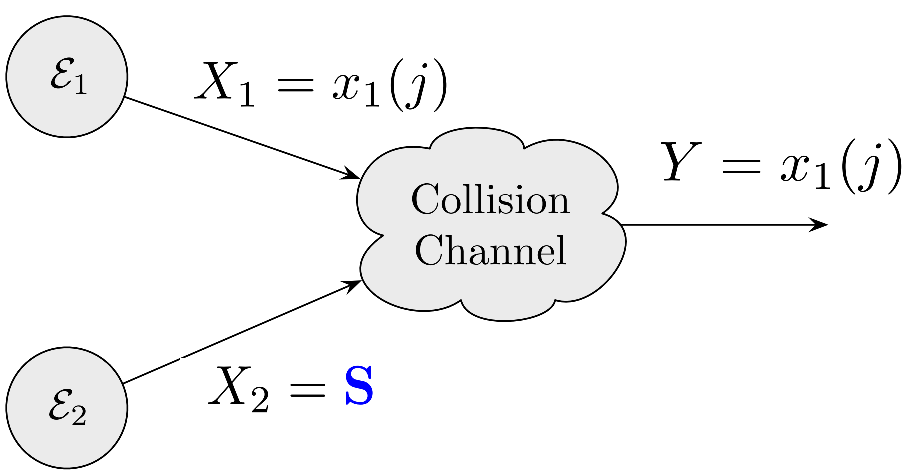
Collision!

The implicit binary noiseless channel

$$U_i = \begin{cases} 1 & \text{if} \quad X_i \neq \mathbf{S} \\ 0 & \text{if} \quad X_i = \mathbf{S} \end{cases}$$

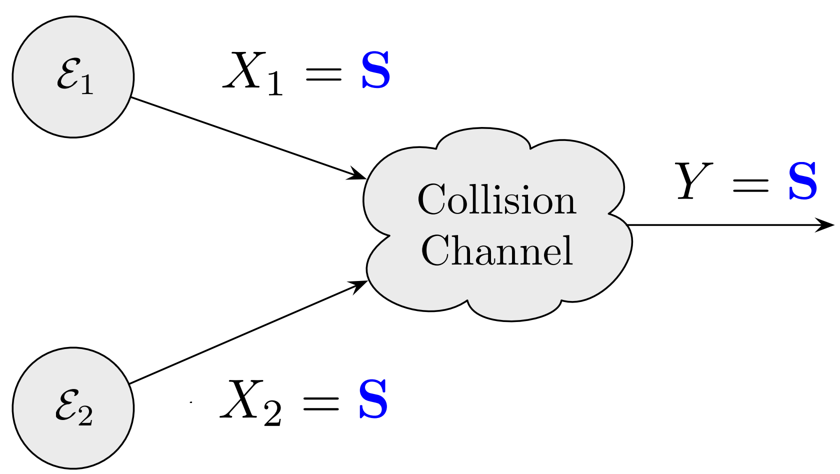
transmit

do not transmit



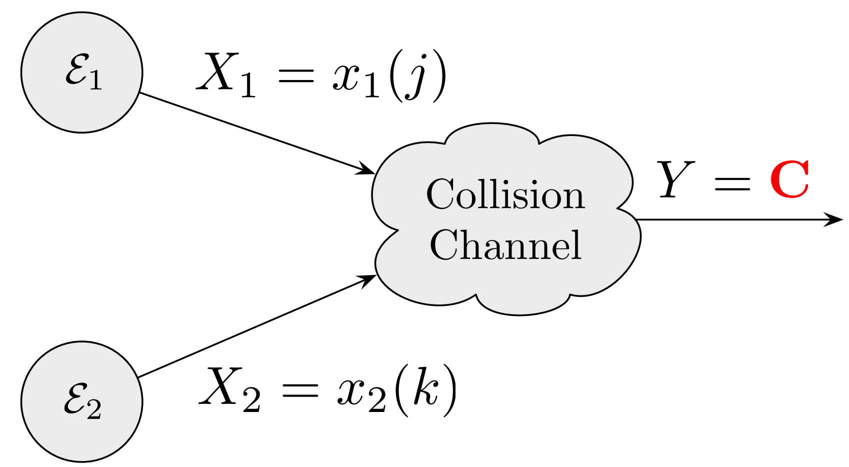
Success

$U_1 = 1, U_2 = 0$



Silence

$U_1 = 0, U_2 = 0$



Collision

$U_1 = 1, U_2 = 1$

From the channel output we can always recover U_1 and U_2

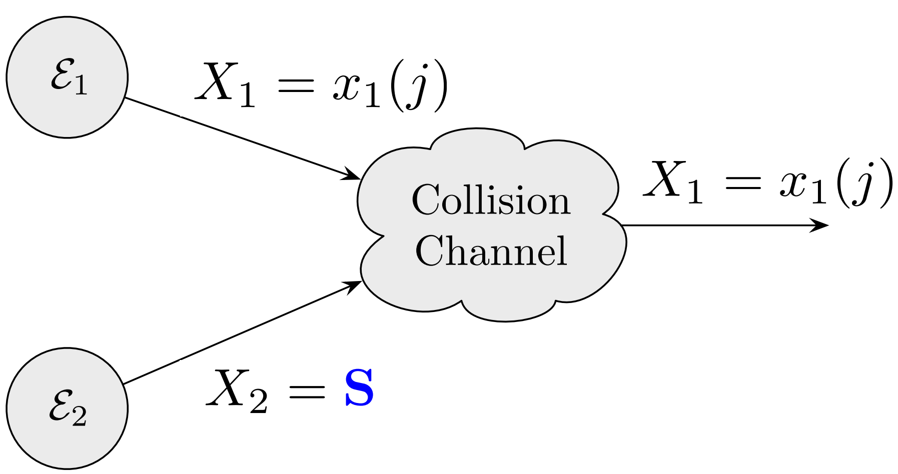
$$H(U_1, U_2 | Y) = 0$$

1. **Dhulipala, Fragouli and Orlitsky, "Silence-based communication",** ***IEEE Trans. IT*** **2010**

The implicit binary noiseless channel

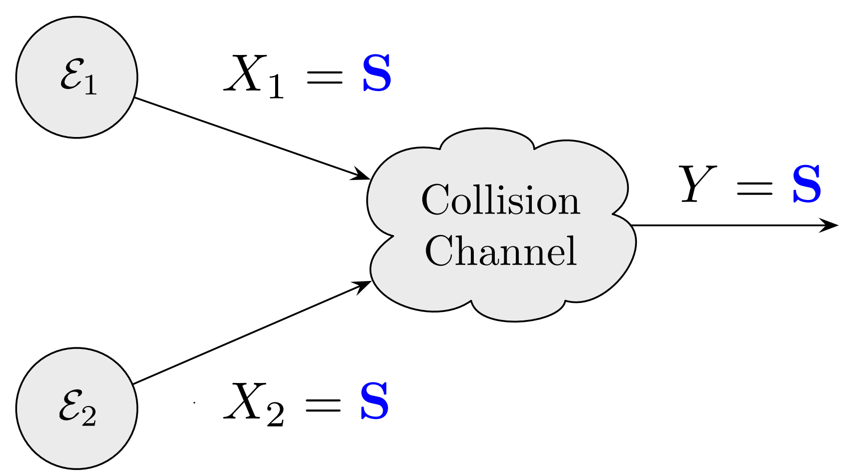
Each user sends at least

$$H(U_i) \quad \text{bits/channel use}$$



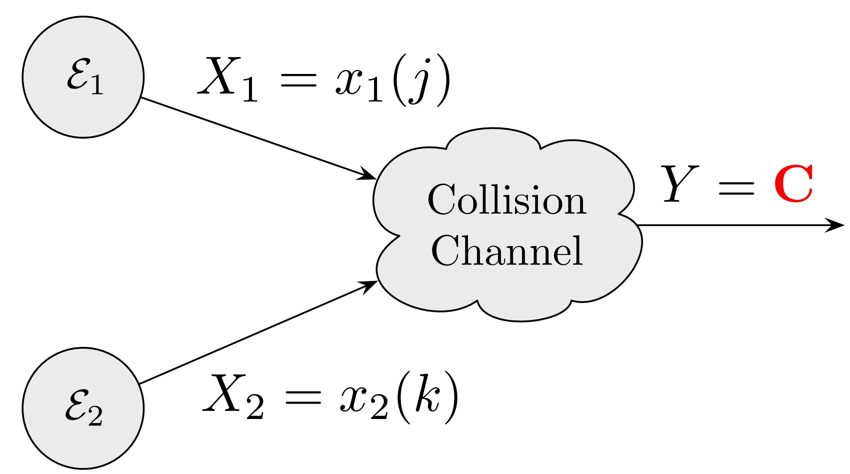
Success

$$U_1 = 1, U_2 = 0$$



Silence

$$U_1 = 0, U_2 = 0$$



Collision

$$U_1 = 1, U_2 = 1$$

Capacity of the collision channel

Theorem 1

The capacity region of the Collision MAC is the convex hull of (R_1, R_2) satisfying

$$R_1 \leq H_b(\theta_1) + \log q_1 \cdot (1-\theta_1) \cdot \theta_2$$
$$R_2 \leq H_b(\theta_2) + \log q_2 \cdot (1-\theta_2) \cdot \theta_1$$

θ_i is the probability of user i being **silent**

$\log q_i$ is number of payload bits of user i

$$R_i \leq \overbrace{H_b(\theta_i)}^{\text{implicit channel}} + \overbrace{\log q_i \cdot \underbrace{(1-\theta_i) \cdot \theta_j}_{\Pr(\text{success for } \mathcal{E}_i)}}^{\text{explicit channel}}$$

$$\theta_i = \Pr(X_i = \mathbf{S})$$

$$H_b(\theta) = \theta \log \frac{1}{\theta} + (1-\theta) \log \frac{1}{1-\theta}$$

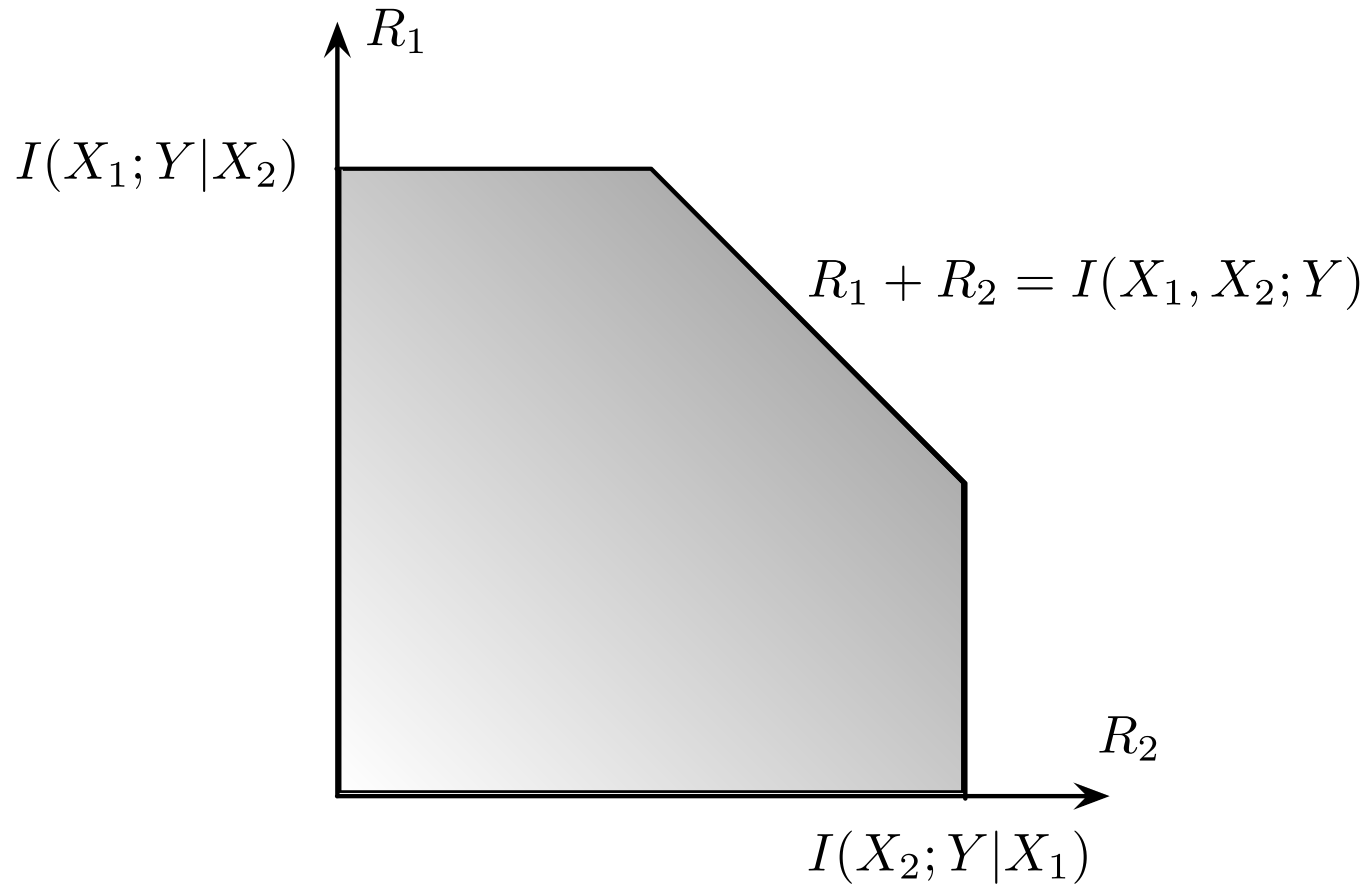
binary entropy function

Sketch of proof

The capacity region of a DMAC is the convex-hull of (R_1, R_2) satisfying:

$$R_1 \le I(X_1; Y | X_2)$$
$$R_2 \le I(X_2; Y | X_1)$$
$$R_1 + R_2 \le I(X_1, X_2; Y)$$

$$P_{X_1 X_2} = P_{X_1} P_{X_2}$$



2. Ahlswede, “Multi-way communication channels,” ISIT 1971
3. Liao, “Multiple access channels,” PhD thesis 1972

Sketch of proof

Lemma 1

The capacity achieving distributions are **uniform** on the **non-silent** symbols

$$I(X_i; Y | X_j) = H(U_i) + H(X_i | U_i = 1) \times \Pr(U_i = 1) \cdot \Pr(U_j = 0)$$

$$\Big\downarrow \; H(X_i | U_i = 1) \leq \log q_i$$

$$I(X_i; Y | X_j) \leq H(U_i) + \log q_i \cdot \Pr(U_i = 1) \cdot \Pr(U_j = 0)$$

Sketch of proof

Lemma 2

The sum-rate inequality is **redundant**

$$
\begin{aligned}
I(X_1, X_2; Y) &\overset{(a)}{=} H(Y) \\
&\overset{(b)}{=} H(Y, U_1, U_2) \\
&\overset{(c)}{=} H(U_1) + H(U_2) + H(Y|U_1, U_2)
\end{aligned}
$$

(a) The channel is deterministic
(b) Implicit channel property
(c) Chain rule

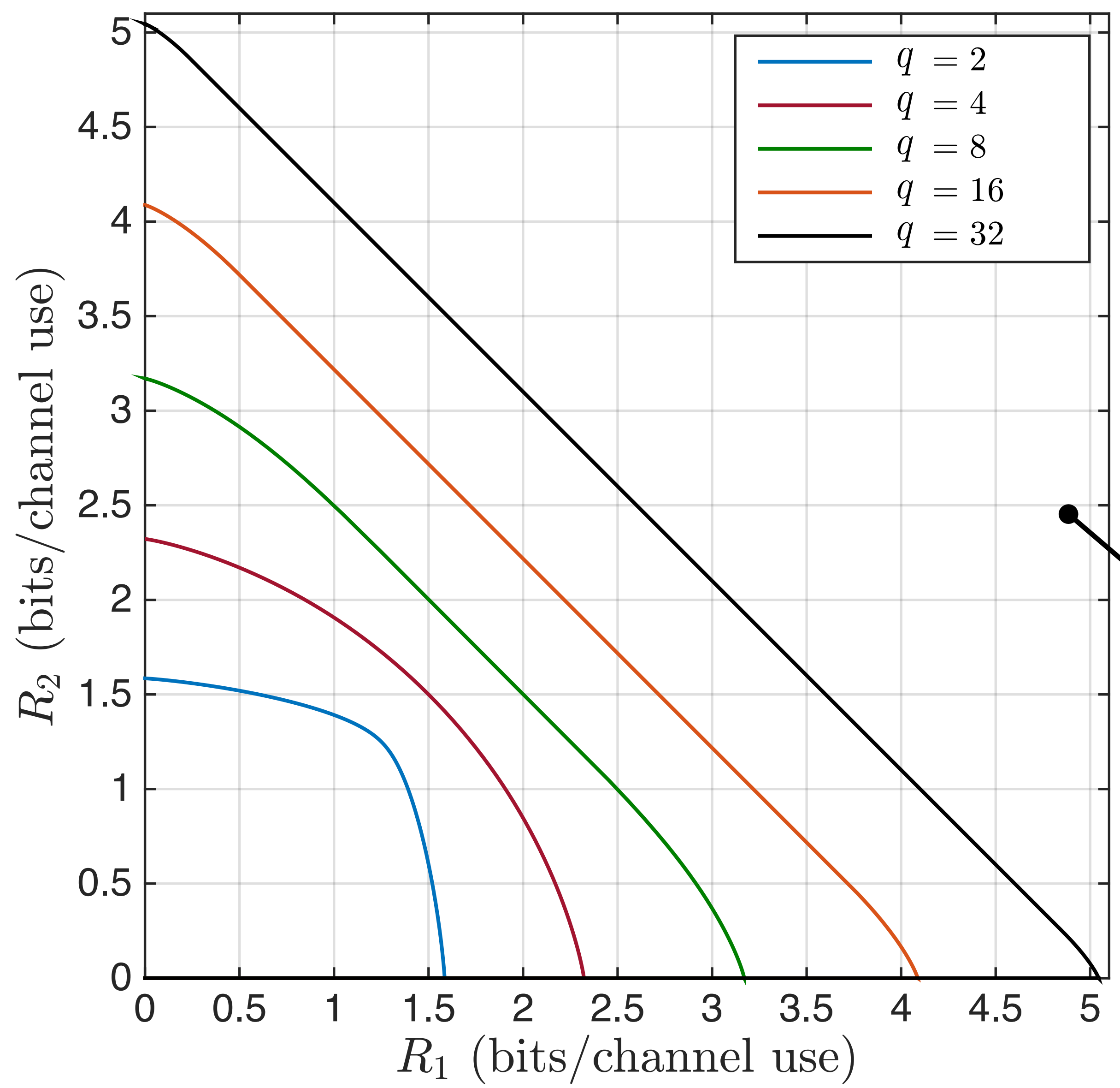
$$
\begin{aligned}
H(Y|U_1, U_2) = H(X_1|U_1 = 1)\Pr(U_1 = 1)\Pr(U_2 = 0) \\
+ H(X_2|U_2 = 1)\Pr(U_2 = 1)\Pr(U_1 = 0)
\end{aligned}
$$

$$
I(X_1, X_2; Y) = I(X_1; Y|X_2) + I(X_2; Y|X_1)
$$

■

Computing the capacity region

$$\begin{array}{ll} \text{maximize} & \mu\Big[H_b(\theta_1) + \log q_1 \cdot (1-\theta_1)\cdot\theta_2\Big] + (1-\mu)\Big[H_b(\theta_2) + \log q_2 \cdot (1-\theta_2)\cdot\theta_1\Big] \\ \text{subject to} & 0 \le \theta_i \le 1, \quad i \in \{1,2\} \end{array}$$

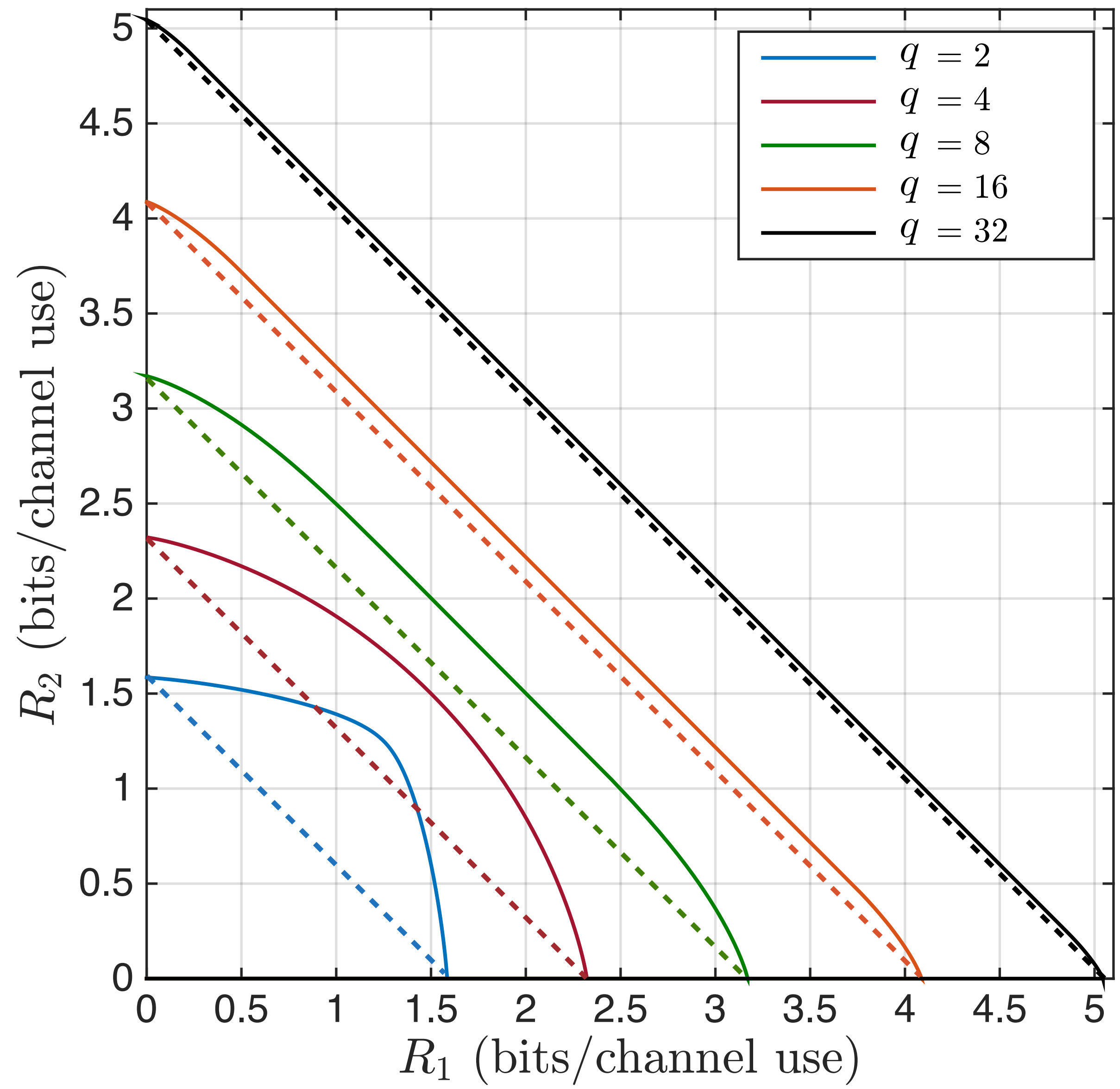


symmetric case:

$q_1 = q_2 = q$

4. **Calvo, Palomar, Fonollosa, and Vidal, "On the computation of the capacity of the discrete MAC", *TCOM* 2010**

Capacity region of the collision channel



Time-sharing inner bound

Small alphabets: non-trivial use of silence and collision symbols

Large alphabets: time-sharing approaches the capacity

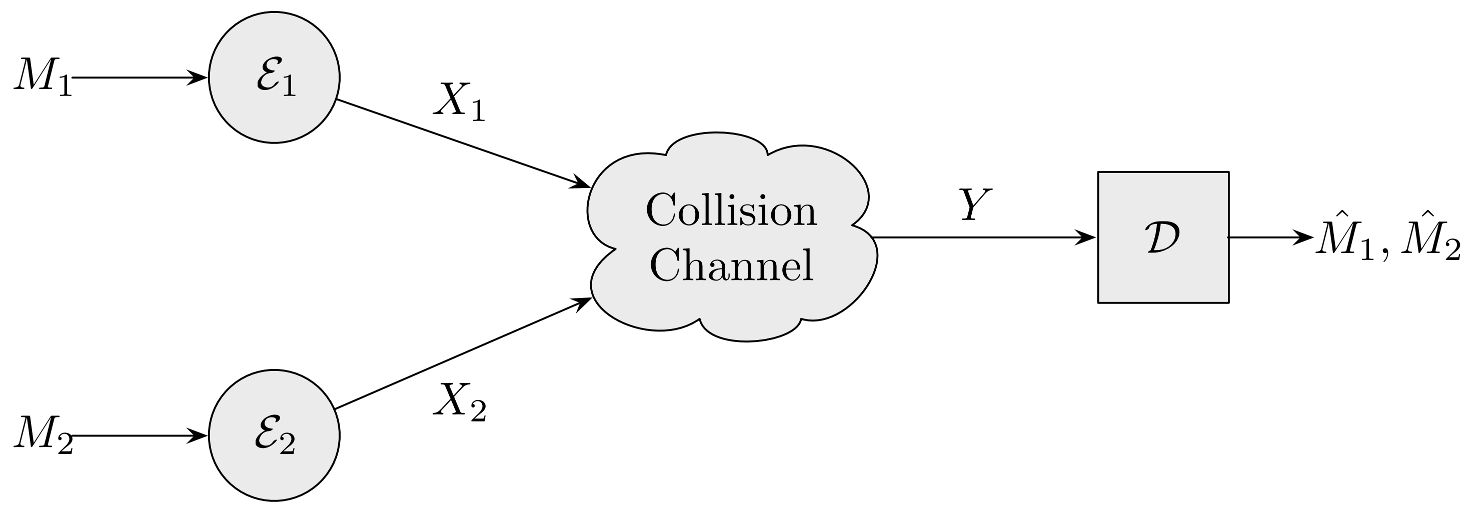
A mutual information game

$$R_1 \leq I(X_1; Y | X_2)$$

$$R_2 \leq I(X_2; Y | X_1)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y)$$ ⟵ **redundant**

Each user maximizes its own maximum achievable rate



Actions

Player 1 chooses P_{X_1}

Player 2 chooses P_{X_2}

Pay-offs

$$\mathcal{J}_1(P_{X_1}, P_{X_2}) = I(X_1; Y | X_2)$$

$$\mathcal{J}_2(P_{X_1}, P_{X_2}) = I(X_2; Y | X_1)$$

5. Berry and Tse, "Shannon Meets Nash on the Interference Channel", *IEEE Trans. IT* 2011
6. Médard, "Capacity of correlated jamming channels", *Allerton* 1997
7. Shaffie and Ulukus, "Mutual Information Games in Multiuser Channels with Corr. Jamming", *IEEE Trans. IT* 2009

A mutual information game

Nash-equilibrium solution

$$\mathcal{J}_1(P^*_{X_1}, P^*_{X_2}) \geq \mathcal{J}_1(P_{X_1}, P^*_{X_2}), \quad \forall P_{X_1}$$
$$\mathcal{J}_2(P^*_{X_1}, P^*_{X_2}) \geq \mathcal{J}_2(P^*_{X_1}, P_{X_2}), \quad \forall P_{X_2}$$

Does a Nash-equilibrium exist?

Theorem 2

The Nash-equilibrium solution **exists** and is **unique**

Sketch of proof

Fixing P_{X_2}

$$\mathcal{J}_1(P_{X_1}, P_{X_2}) \leq H(U_1) + \log q_1 \cdot \Pr(U_1 = 1) \cdot \Pr(U_2 = 0)$$

$$\Updownarrow$$

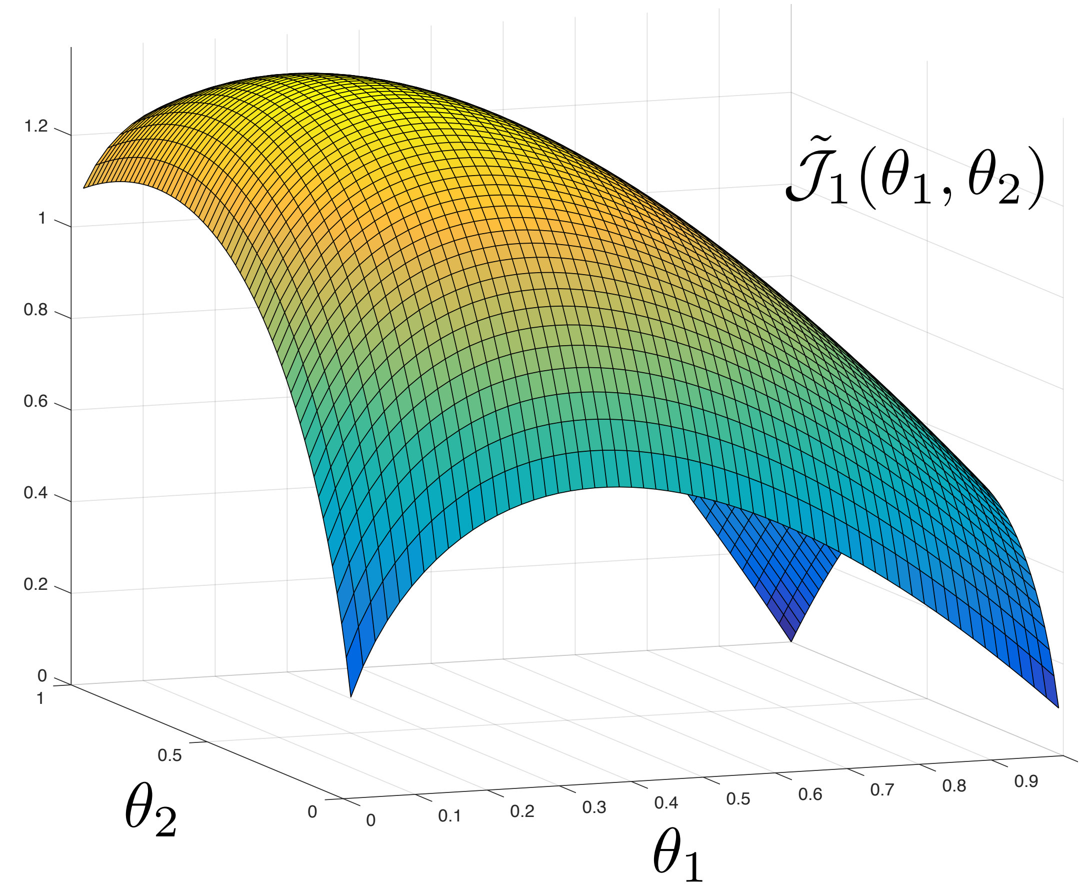
$$P^*_{X_i}(\theta_i) = \left[\theta_i, \frac{1-\theta_i}{q_i}, \cdots, \frac{1-\theta_i}{q_i}\right]$$

$$\tilde{\mathcal{J}}_1(\theta_1, \theta_2) \triangleq \mathcal{J}_1\left(P^*_{X_1}(\theta_1), P^*_{X_2}(\theta_2)\right)$$

$$\tilde{\mathcal{J}}_1(\theta_1, \theta_2) = H_b(\theta_1) + \log q_1 \cdot \theta_1 \cdot (1 - \theta_2)$$

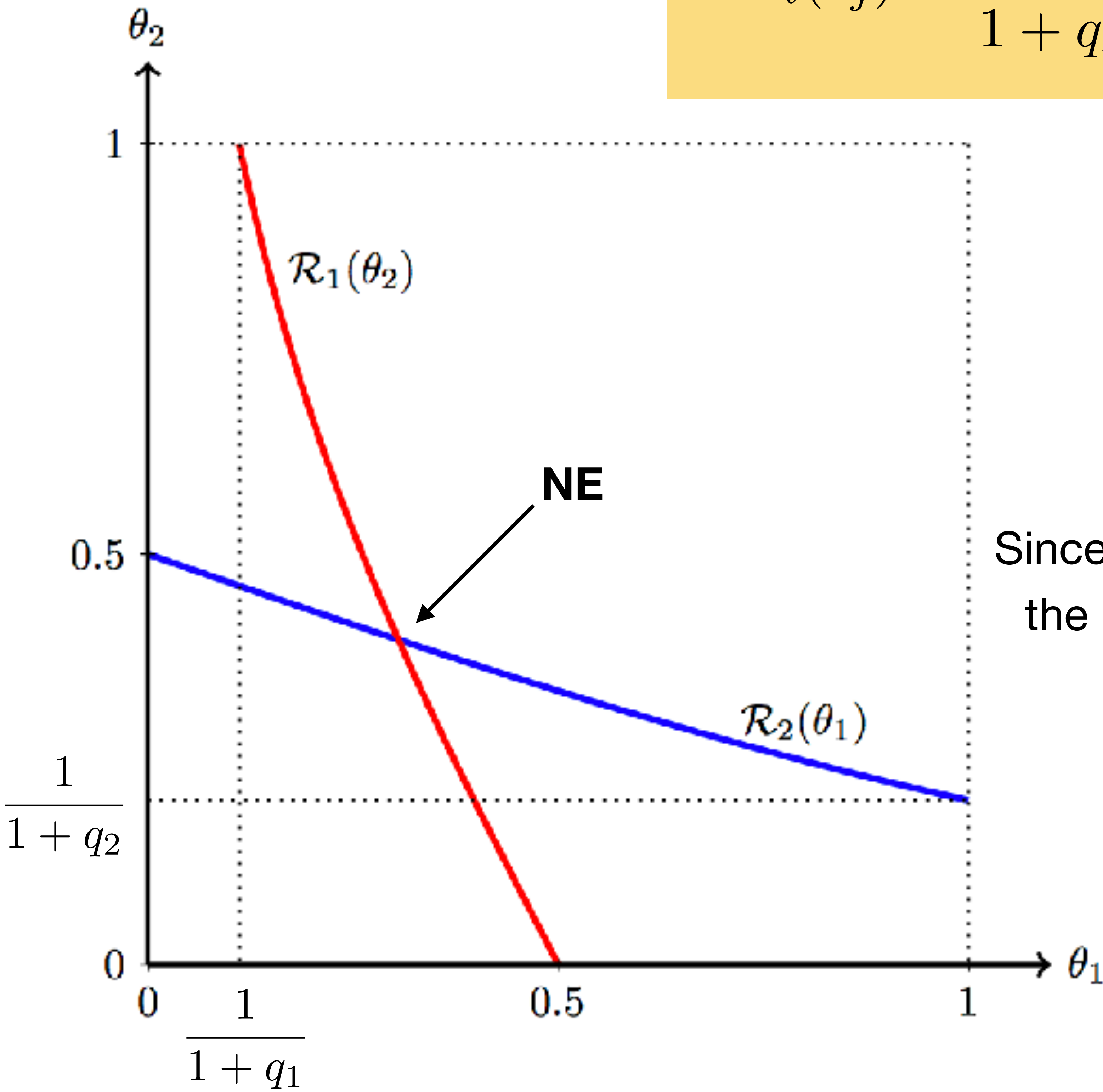
For every fixed $\theta_2 \in [0, 1]$

$\tilde{\mathcal{J}}_1(\theta_1, \theta_2)$ is strictly concave on θ_1



Reaction curves

$$\mathcal{R}_i(\theta_j) = \frac{1}{1 + q_i^{\theta_j}}$$



Every Nash-equilibrium
lies on **both reaction curves**

Since they always **intersect** at a single point,
the Nash-equilibrium **exists** and is **unique**

■

Summary & future work

1. Fundamental limits of communication over the collision channel

2. Trade-off between implicit channel vs. explicit channel

3. Operation at the boundary requires centralized design (cooperation)

4. Selfish behavior leads to a game

1. Capacity of other types of collision channel:

- **i. feedback**
- **ii. type II** $\boxed{\mathbf{S} = \mathbf{C}}$
- **iii. capture**

2. Is the Nash-equilibrium stable? Price-of-Anarchy?

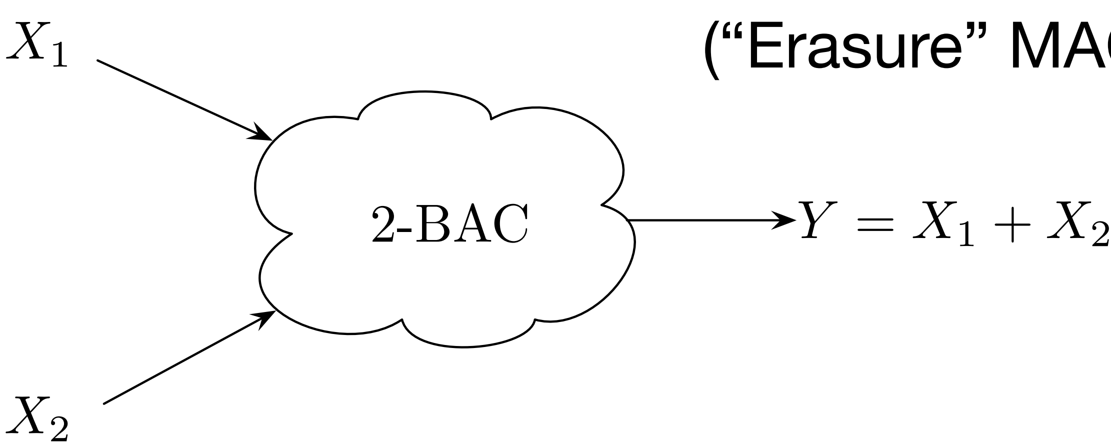
3. Are there practical codes for the collision channel?

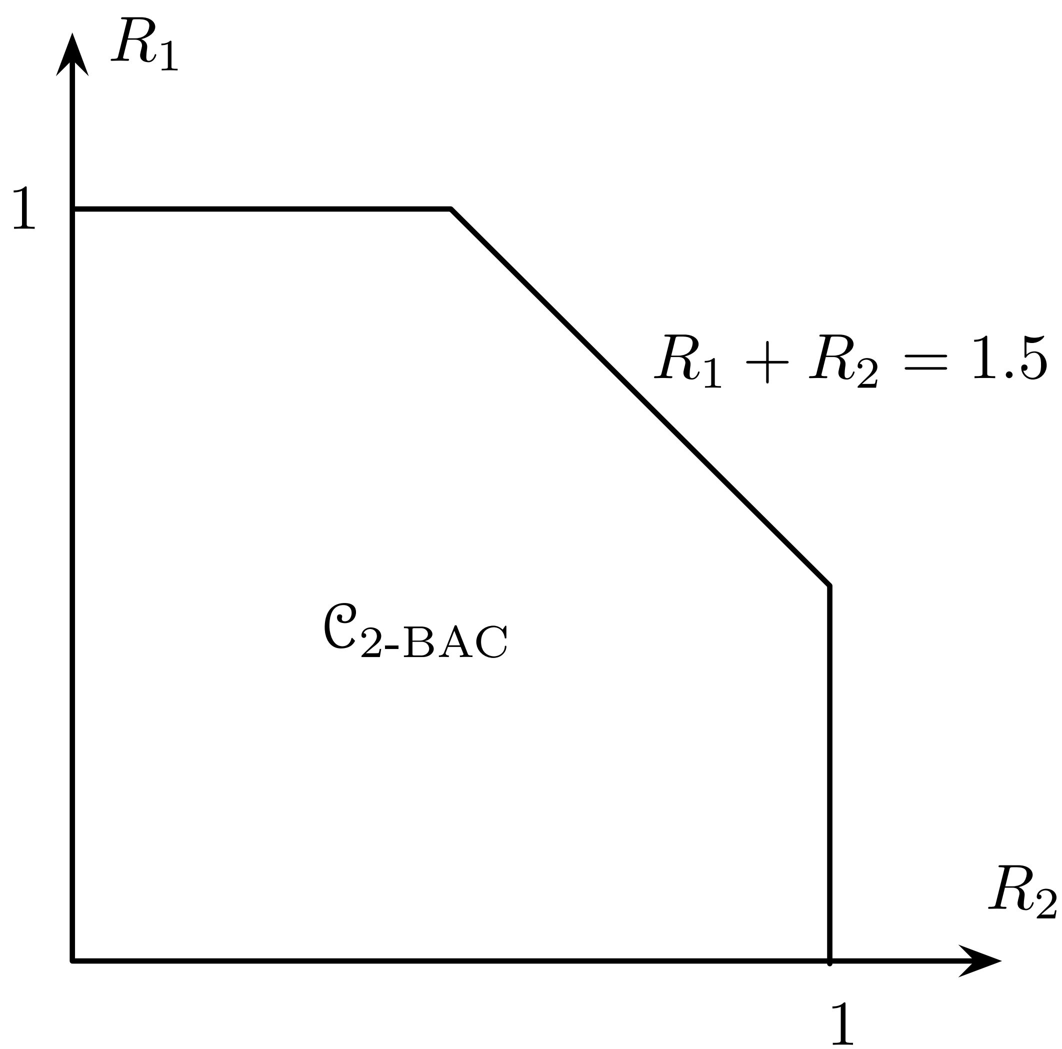
Appendix

Other deterministic MACs

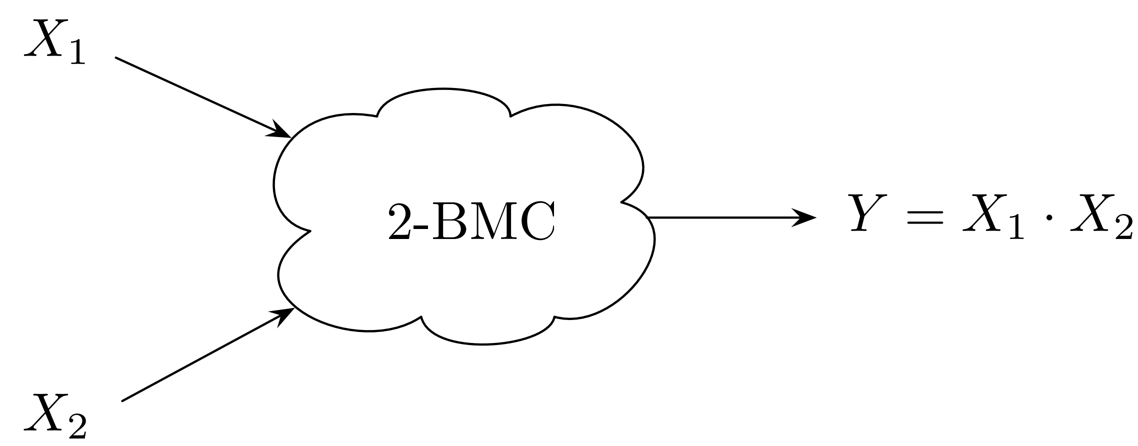
Binary Adder Channel

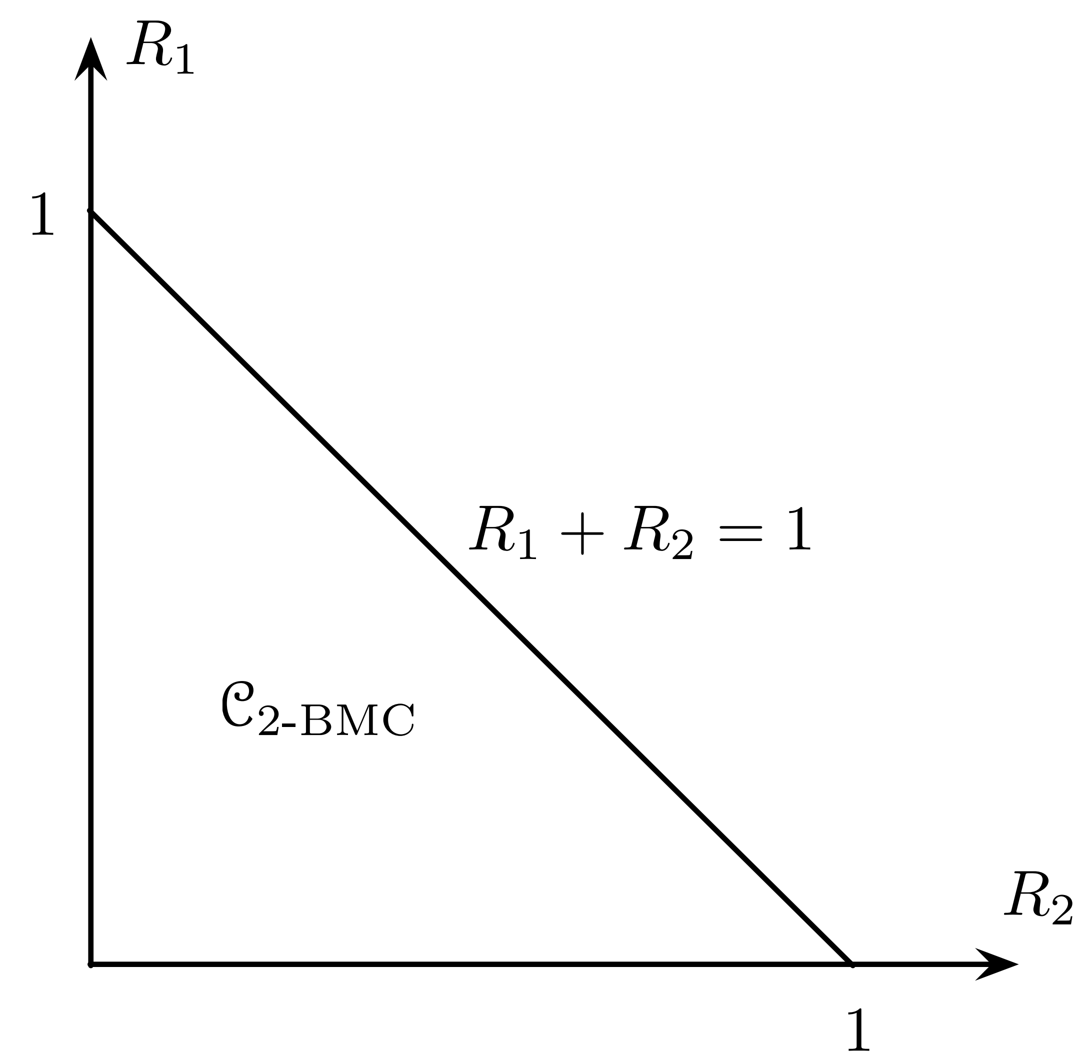
(“Erasure” MAC)





Binary Multiplier Channel





6. El Gammal and Kim, *Network Information Theory*

Computing the capacity region

$$\begin{array}{ll} \text{maximize} & \mu R_1 + (1-\mu) R_2 \\ \text{subject to} & R_1 \le I(X_1; Y | X_2) \\ & R_2 \le I(X_2; Y | X_1) \\ & R_1 + R_2 \le I(X_1, X_2; Y) \end{array}$$

with variables $R_1, R_2, P_{X_1}, P_{X_2}$

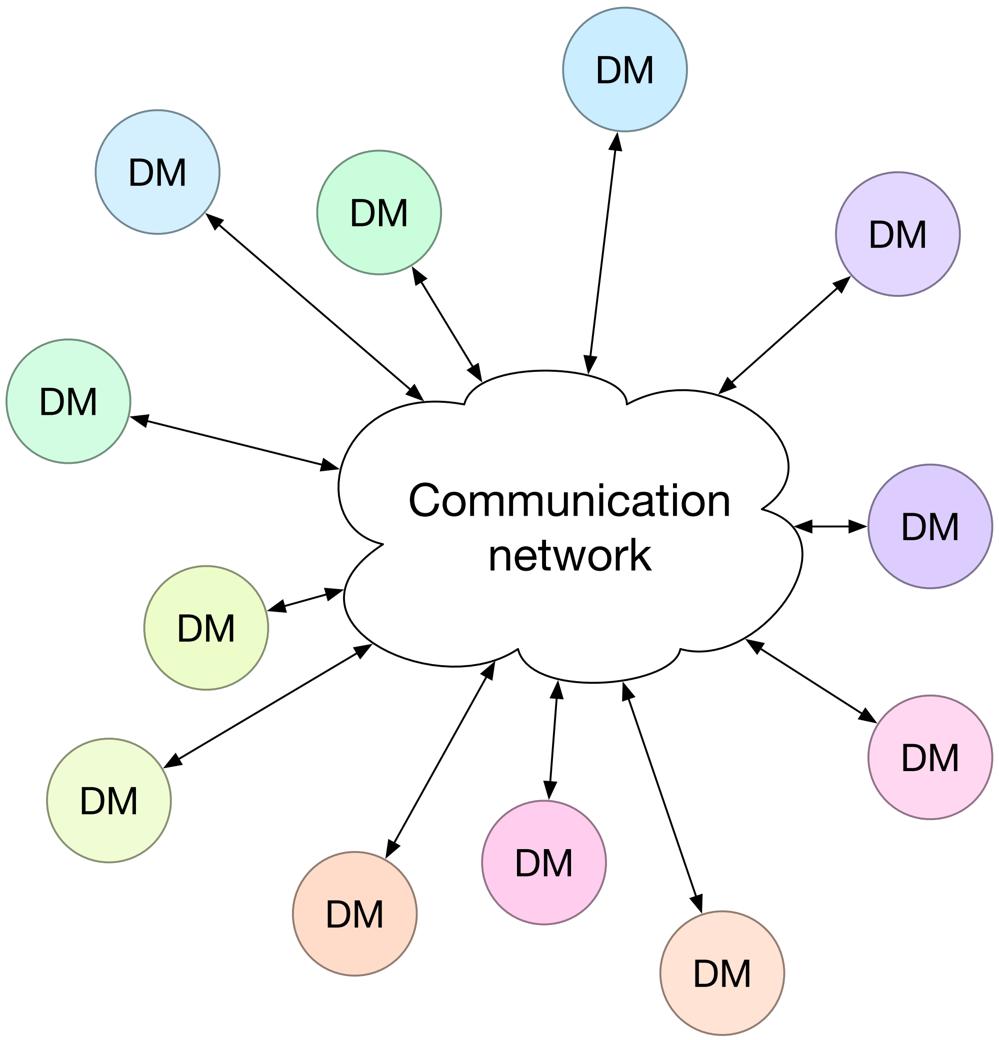
$\downarrow$ **Theorem 1**

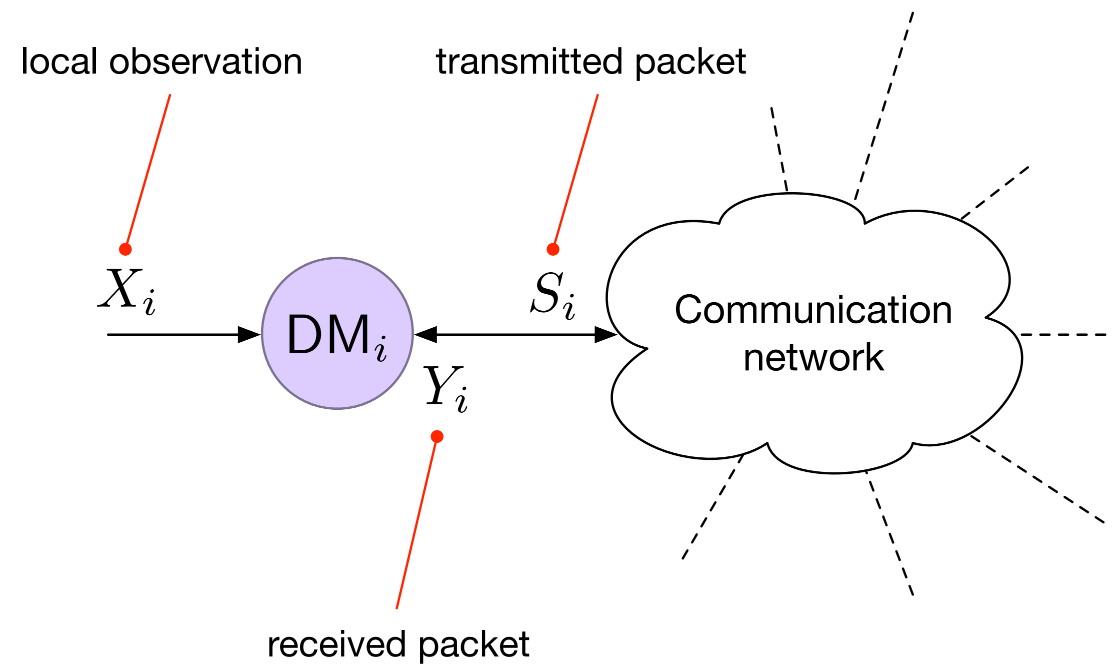
$$\begin{array}{ll} \text{maximize} & \mu\Big[H_b(\theta_1) + \log q_1 \cdot (1-\theta_1) \cdot \theta_2\Big] + (1-\mu)\Big[H_b(\theta_2) + \log q_2 \cdot (1-\theta_2) \cdot \theta_1\Big] \\ \text{subject to} & 0 \le \theta_i \le 1, \quad i \in \{1, 2\} \end{array}$$

Non-convex, in general

6. Calvo, Palomar, Fonollosa, and Vidal, "On the computation of the capacity of the discrete MAC", *TCOM* 2010

Networked decision systems





Many applications

1. **Networked control**
2. **Remote estimation**
3. **Sensor networks**
4. **Robotic networks**

Many challenges

Communication is imperfect:
Delays, noise, quantization, congestion, packet drops, connectivity and
packet collisions

1. **Vasconcelos and Martins, "Optimal estimation over the collision channel", *IEEE TAC* 2017**
2. **Vasconcelos and Martins, "Remote estimation games over the shared networks", *Allerton* 2015**