Estimation over the Collision Channel

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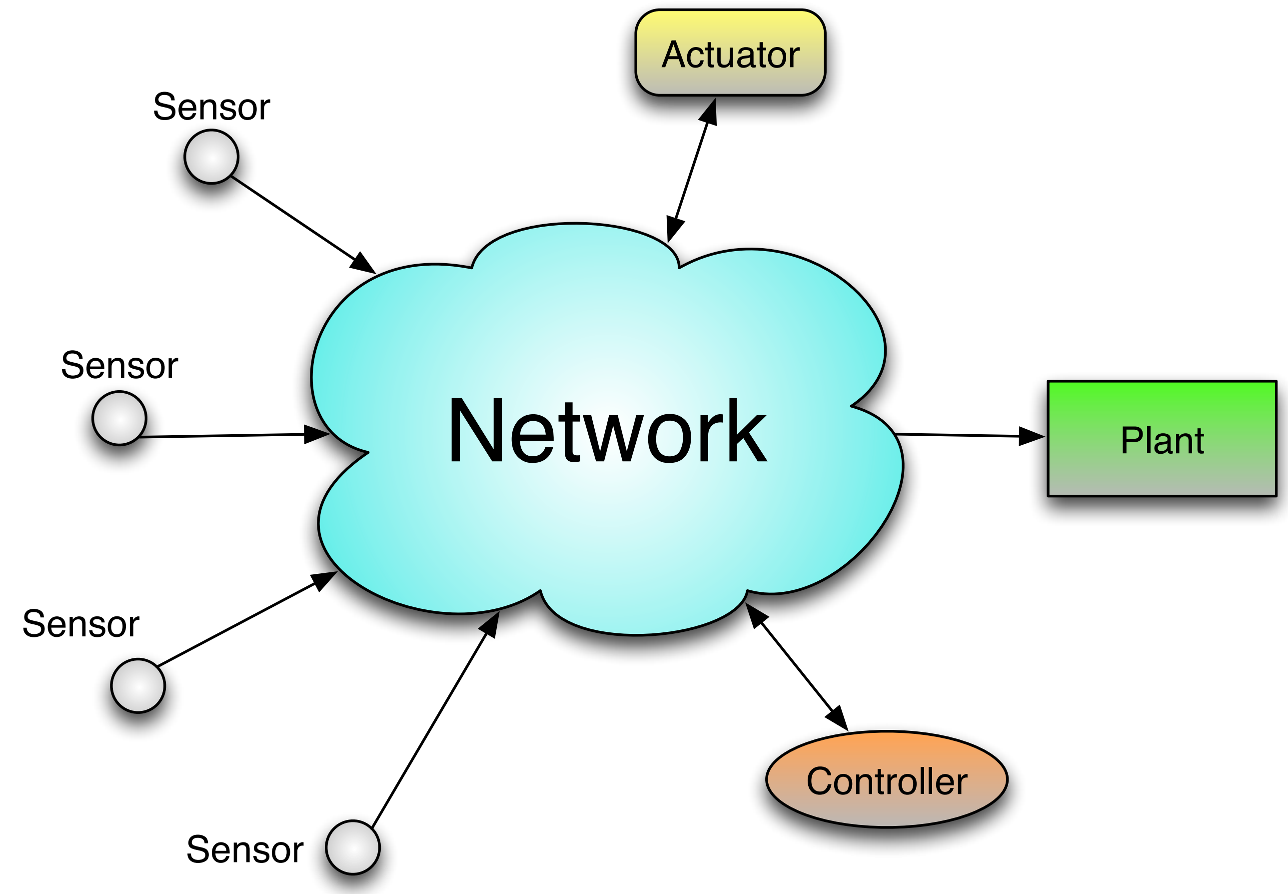
Motivation

Context

- Networked control systems
- Decision Makers cooperating to achieve a common goal

Network models

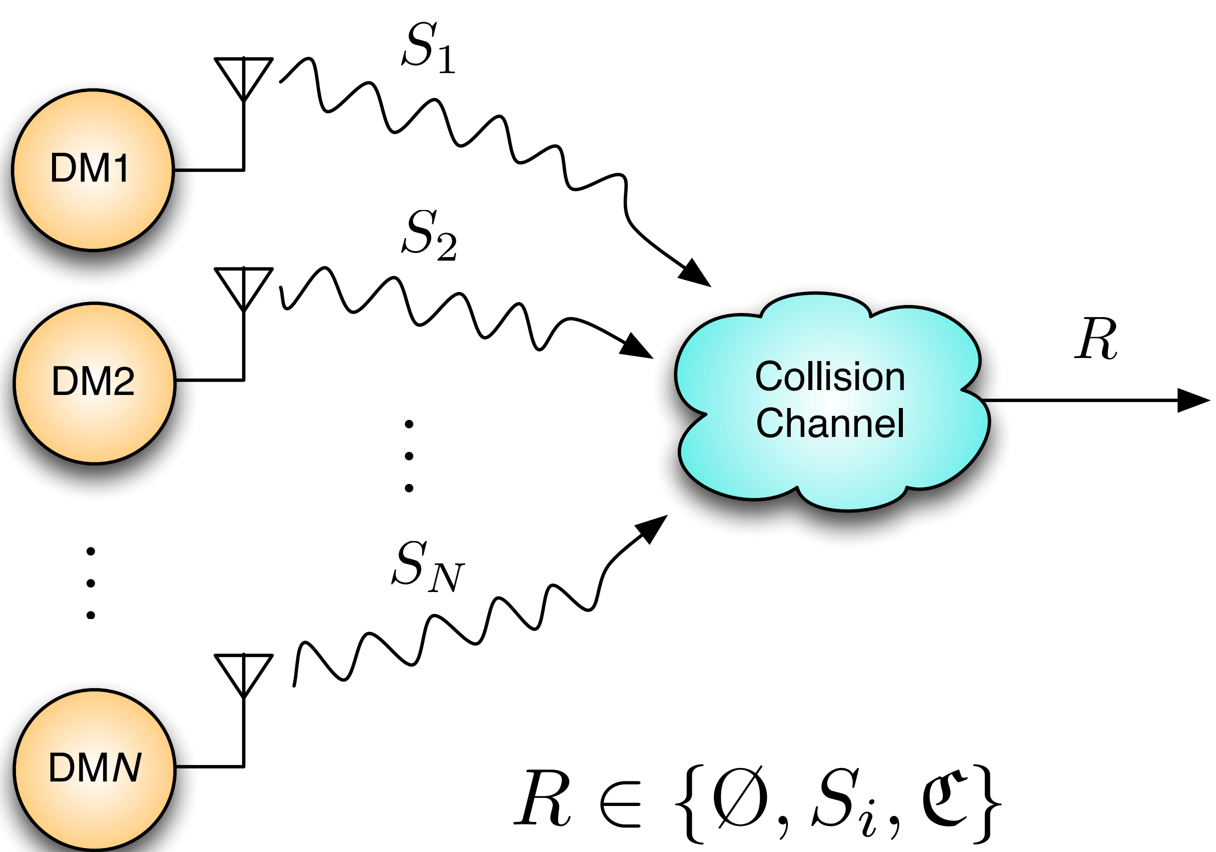
- Incomplete graphs
- Rate-limited point-to-point channels
- Additive White Gaussian Noise
- Analog Erasure channel



What about interference?

Interference

- Multiple agents sharing a communication medium
- Physical layer: Multiple Access Channel
- MAC/Network layer: Collision Channel

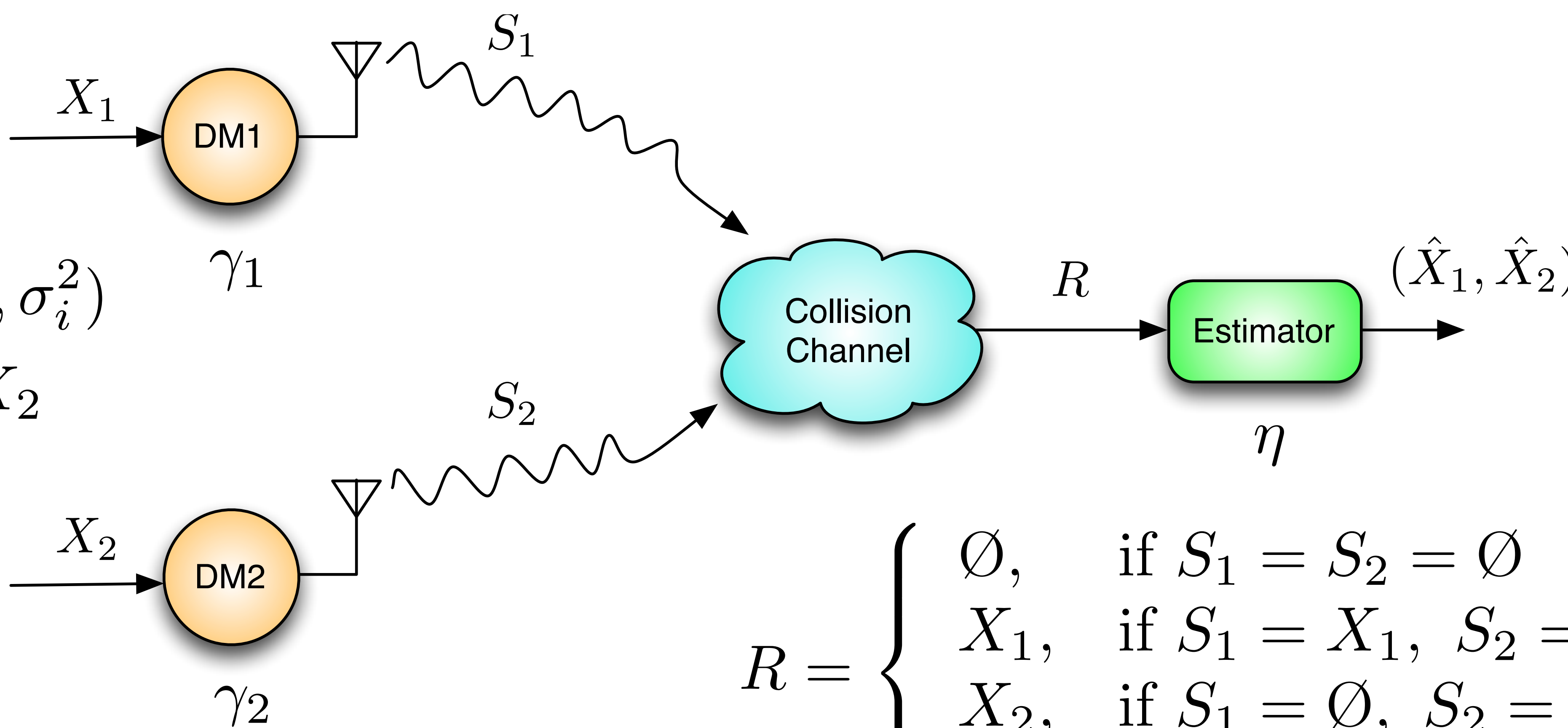


$$R \in \{\emptyset, S_i, \mathfrak{C}\}$$

Channel Model

- DM chooses to transmit or not
- Collision when two or more DMs transmit
- Simplest model for interference

Problem Statement



$$X_i \sim \mathcal{N}(0, \sigma_i^2)$$

$$X_1 \perp\!\!\!\perp X_2$$

$$R = \begin{cases} \emptyset, & \text{if } S_1 = S_2 = \emptyset \\ X_1, & \text{if } S_1 = X_1,\ S_2 = \emptyset \\ X_2, & \text{if } S_1 = \emptyset,\ S_2 = X_2 \\ \mathfrak{C}, & \text{if } S_1 = X_1,\ S_2 = X_2 \end{cases}$$

$$U_i = \gamma_i(X_i) \in \{0, 1\}$$

$$S_i = \begin{cases} X_i, & \text{if } U_i = 1 \\ \emptyset, & \text{if } U_i = 0 \end{cases}$$

$$\eta(R) = (\eta_1(R), \eta_2(R))$$

$$(\hat{X}_1, \hat{X}_2) = \eta(R)$$

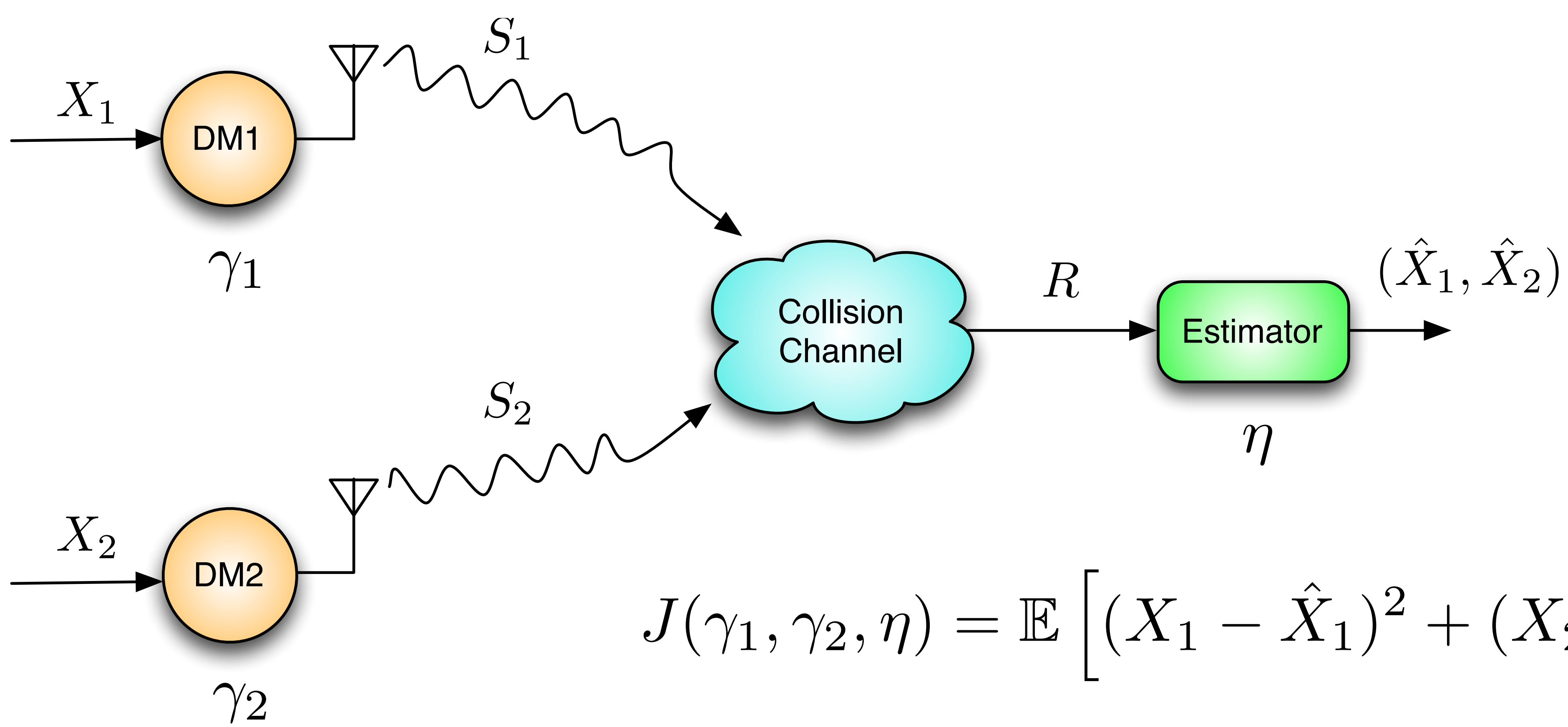
Jointly minimize the following cost functional

$$J(\gamma_1, \gamma_2, \eta) = \mathbb{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right]$$

Problem Statement

Why is this a relevant problem?

- New channel model for distributed estimation/control problems
- Team decision problem (non-convex in general)
- Captures the effects of an important network limitation



$$J(\gamma_1, \gamma_2, \eta) = \mathbb{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right]$$

Our main results

1. Relationship with remote estimation with communication costs
2. Optimality of threshold policies using infinite dimensional LP
3. Connection with quantization theory
4. Propose an iterative algorithm: Modified Lloyd-Max

Part 1. Remote Estimation with Communication Costs
Subproblem

- Every transmission made by DM1 potentially causes collisions for DM2 (and vice-versa)
- For a fixed (γ_2, η_2), DM1 must pay a cost per transmission attempt

$$J = \mathsf{E}[(X_1 - \hat{X}_1)^2] + \left(\mathsf{E}\left[(X_2 - \hat{X}_2)^2 | U_1 = 1\right] - \mathsf{E}\left[(X_2 - \hat{X}_2)^2 | U_1 = 0\right]\right) \Pr(U_1 = 1)$$
$$+ \mathsf{E}[(X_2 - \hat{X}_2)^2 | U_1 = 0]$$

Lemma: ***For fixed*** (γ_2, η_2) ***s.t.*** $\eta_2(X_2) = X_2$***, we have***

$$\rho_{21} \triangleq \mathbb{E}\left[(X_2 - \hat{X}_2)^2 | U_1 = 1\right] - \left[(X_2 - \hat{X}_2)^2 | U_1 = 0\right] \geq 0$$

Part 1. Remote Estimation with Communication Costs
Subproblem

Proposition: From the perspective of DM1, the problem is equivalent to jointly minimizing the following cost functional

$$J(\gamma_1, \eta_1) = \mathbb{E}\left[(X_1 - \hat{X}_1)^2\right] + \rho_{21}\,\Pr(U_1 = 1)$$

- Absence of collisions: solved by Lipsa & Martins (TAC '11)

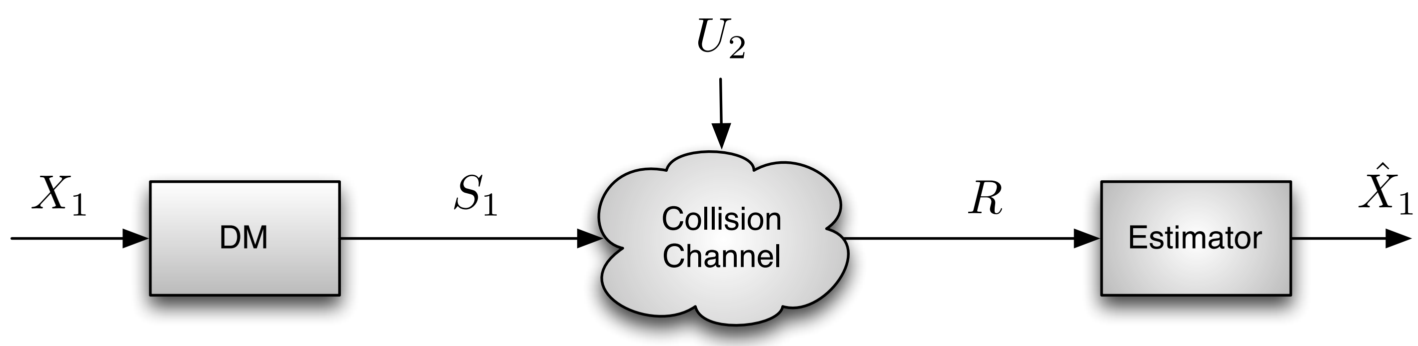
  Solution: symmetric threshold policies

- Presence of collisions:

  Solution: asymmetric threshold policies

Part 2. Optimality of threshold policies

- Fixing the policies of DM2:



- Channel is occupied w.p. $\Pr(U_2 = 1) = \beta,\ U_2 \perp\!\!\!\perp X_1$

- Drop all the subscripts and work in the single DM case.

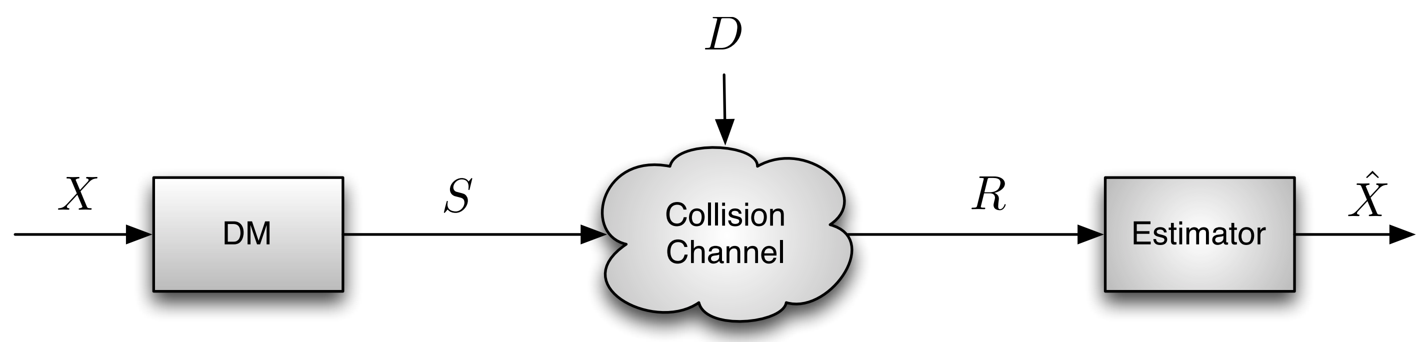
$$D \triangleq U_2$$

$$\rho \triangleq \rho_{21}$$

$$U \triangleq U_1$$

$$(\gamma, \eta) \triangleq (\gamma_1, \eta_1)$$

Part 2. Optimality of threshold policies

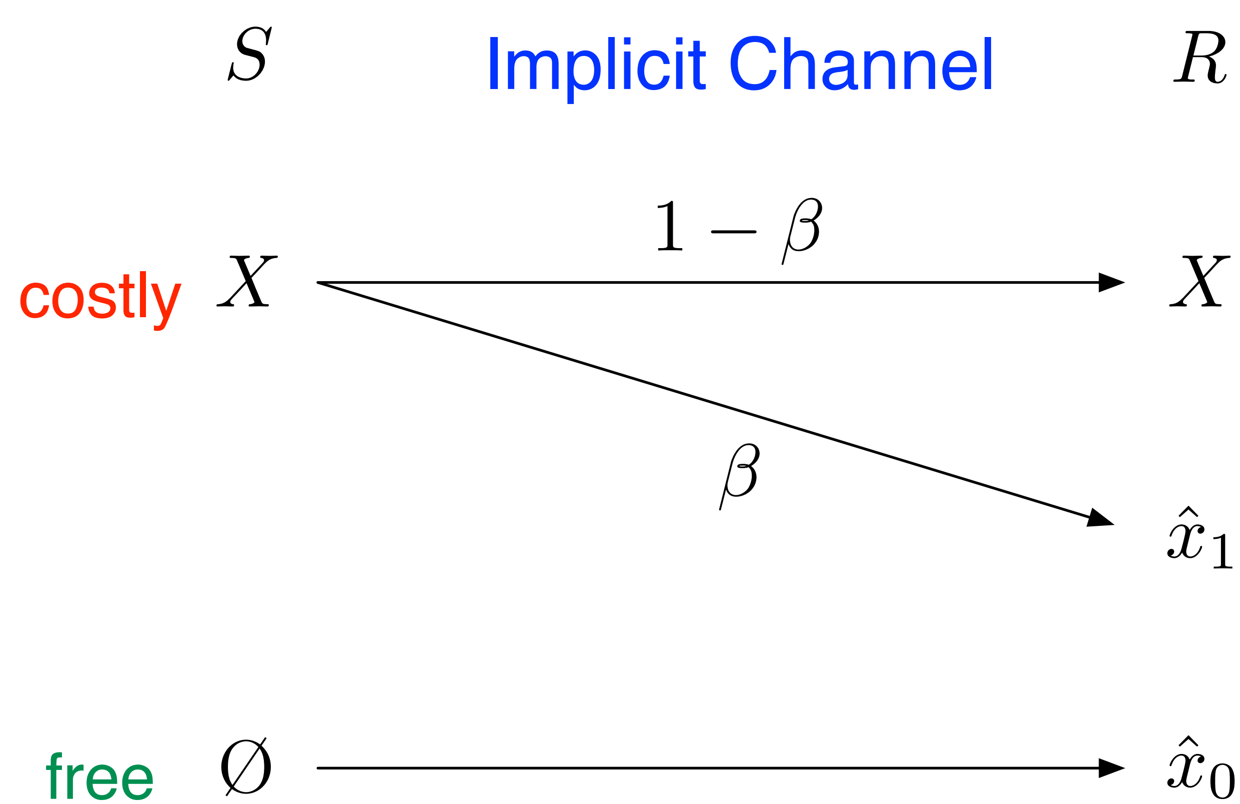


- Channel is occupied w.p. $\Pr(D = 1) = \beta, \;\; D \perp\!\!\!\perp X$

- Given $X = x$, let $U \sim \mathcal{B}(g(x)), \;\; 0 \leq g(x) \leq 1, \;\; \forall x \in \mathbb{R}$

randomized policy

- $U = 0 \quad \Rightarrow \hat{X} = \eta^*(\emptyset) \triangleq \hat{x}_0$
- $(U = 1, D = 0) \Rightarrow \hat{X} = \eta^*(X) = X$
- $(U = 1, D = 1) \Rightarrow \hat{X} = \eta^*(\mathfrak{C}) \triangleq \hat{x}_1$



Part 2. Optimality of Threshold Policies

- The cost functional becomes

$$J = \mathbb{E}\left[(X-\hat{x}_0)^2|U=0\right]\Pr(U=0) + \mathbb{E}\left[\beta(X-\hat{x}_1)^2+\rho|U=1\right]\Pr(U=1)$$

- For a fixed $g(x)$, the optimal values of $(\hat{x}_0, \hat{x}_1)$ are

$$\hat{x}_0 = \mathbb{E}[X|U=0]$$
$$\hat{x}_1 = \mathbb{E}[X|U=1]$$

- Rewriting the cost functional as a function of $g(x)$

$$J = (1-\beta)\left(1-\mathbb{E}[g(X)]\right)\mathbb{E}[X^2|U=0] - \left[\frac{\left(1-\mathbb{E}[g(X)]\right)^2}{\mathbb{E}[g(X)]}\beta + \left(1-\mathbb{E}[g(X)]\right)\right]\hat{x}_0^2 +$$
$$+\rho\mathbb{E}[g(X)] + \beta\sigma^2$$

Clearly non-convex in $g(x)$!

Part 2. Optimality of Threshold Policies

- For every $\hat{x}_0$, constrain $\mathbb{E}[g(X)] = \alpha$

- Rewriting the cost functional as a function of $g(x)$

$$J(\alpha) = (1-\beta)\,(1-\alpha)\,\mathbb{E}[X^2|U=0] - \left[\frac{(1-\alpha)^2}{\alpha}\beta + (1-\alpha)\right]\hat{x}_0^2 + \rho\alpha + \beta\sigma^2$$

Part 2. Optimality of Threshold Policies

$$\begin{aligned} \underset{g}{\text{minimize}} \quad & \mathbb{E}[X^2 | U = 0] \\ \text{subject to} \quad & \mathbb{E}[g(X)] = \alpha \\ & \mathbb{E}[X | U = 0] = \hat{x}_0 \\ & 0 \leq g(x) \leq 1, \forall x \in \mathbb{R}. \end{aligned}$$

- **Re-parametrizing:** $\mu(x) \triangleq \dfrac{(1 - g(x))\, f_X(x)}{1 - \alpha}$

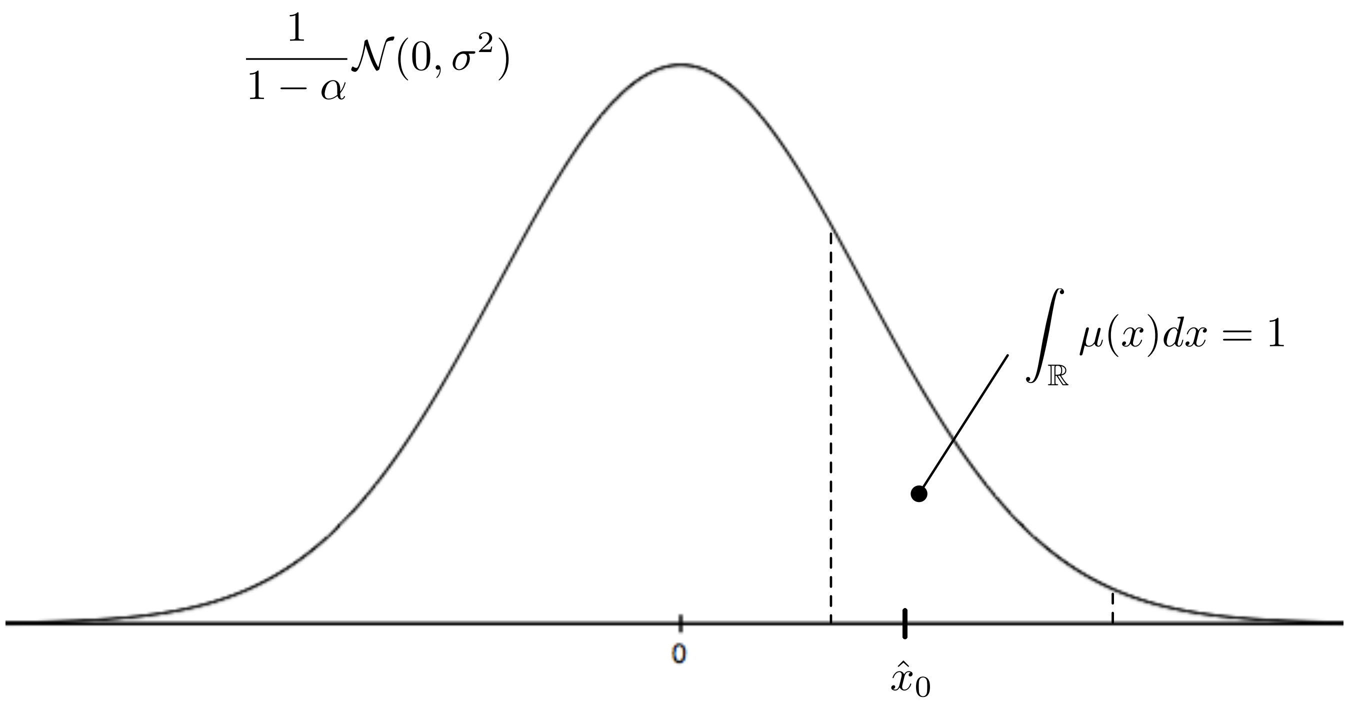
$$\begin{aligned} \underset{\mu \in L^1(\mathbb{R})}{\text{minimize}} \quad & \int_{\mathbb{R}} x^2 \mu(x) dx \\ \text{subject to} \quad & \int_{\mathbb{R}} \mu(x) dx = 1 \\ & \int_{\mathbb{R}} x \mu(x) dx = \hat{x}_0 \\ & \theta_{L^1(\mathbb{R})} \leq \mu \leq \frac{1}{1 - \alpha} \mathcal{N}(0, \sigma^2) \end{aligned}$$

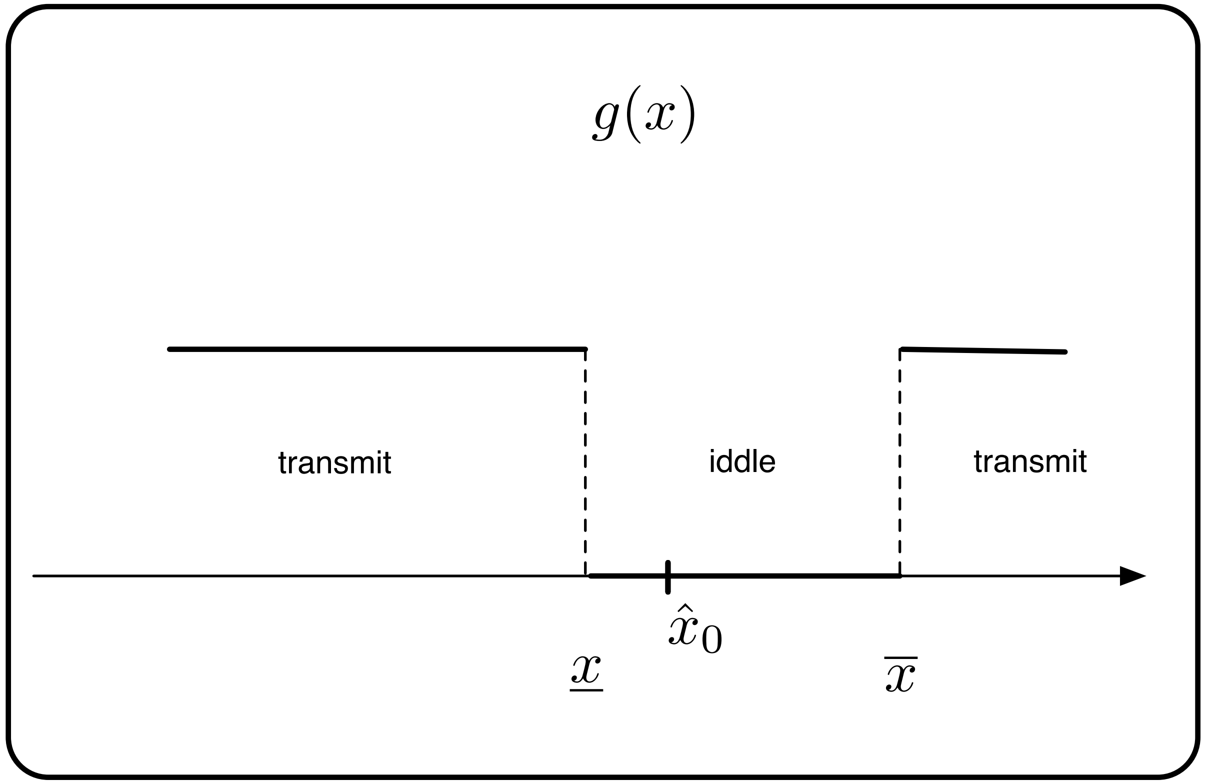
Theorem: The optimal communication policies are of the threshold type.

Part 2. Optimality of Threshold Policies

$$\begin{aligned}
\underset{\mu \in L^1(\mathbb{R})}{\text{minimize}} \quad & \int_{\mathbb{R}} x^2 \mu(x) dx \\
\text{subject to} \quad & \int_{\mathbb{R}} \mu(x) dx = 1 \\
& \int_{\mathbb{R}} x \mu(x) dx = \hat{x}_0 \\
& \theta_{L^1(\mathbb{R})} \leq \mu \leq \frac{1}{1-\alpha} \mathcal{N}(0, \sigma^2)
\end{aligned}$$

- Infinite dimensional LP
- "Entropy minimization with lattice constraints", Borwein et al. (JOTA'94)
- Use duality theory





Part 3. Connection with Quantization Theory

$$J = \mathbb{E}\left[(X-\hat{x}_0)^2|U=0\right]\Pr(U=0) + \mathbb{E}\left[\beta(X-\hat{x}_1)^2+\rho|U=1\right]\Pr(U=1)$$

- Partition $\mathbb{R}$ into $\mathcal{M}$ and $\mathcal{M}_c$ (quantization regions):

$$x \in \mathcal{M}_c \Rightarrow (x-\hat{x}_0)^2$$

$$x \in \mathcal{M} \Rightarrow \beta(x-\hat{x}_1)^2+\rho$$

asymmetric distortion function

- Optimal representation points are the centroids of $\mathcal{M}$ and $\mathcal{M}_c$

$$\Pr(X\in\mathcal{M}_c)\hat{x}_0 = -\Pr(X\in\mathcal{M})\hat{x}_1$$

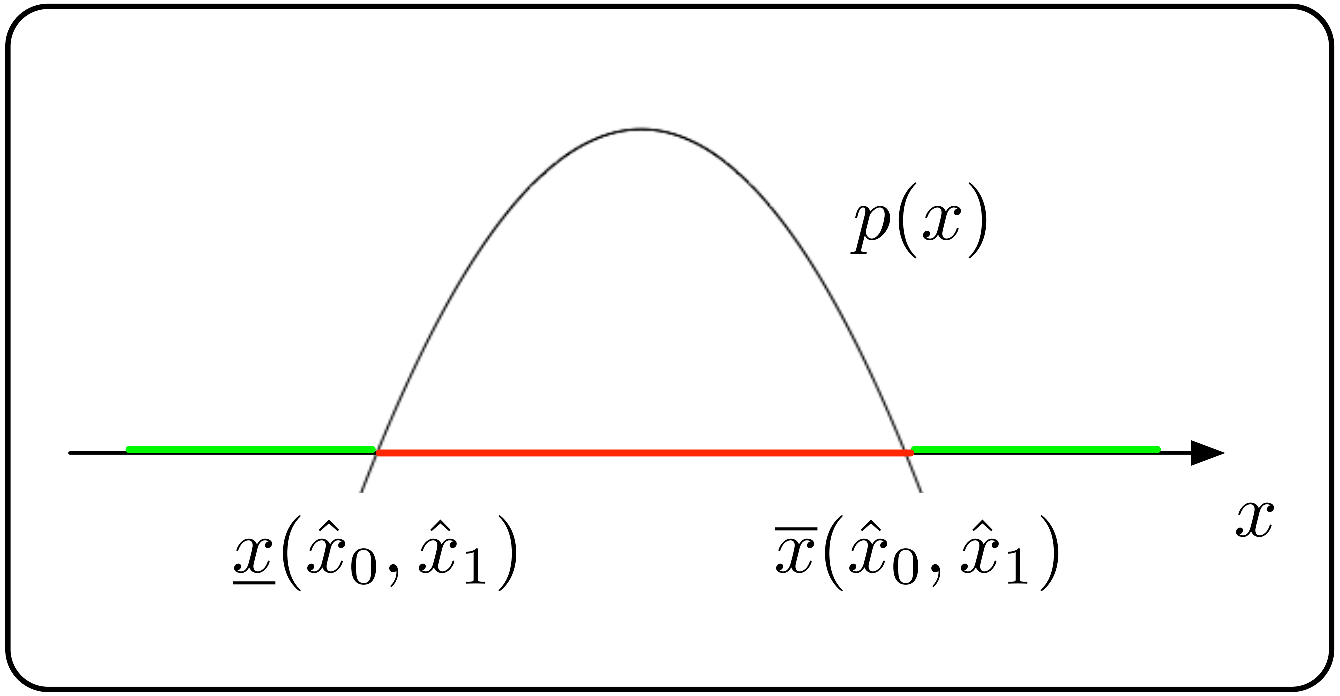
Part 3. Connection with Quantization Theory

$$J(\hat{x}_0, \hat{x}_1, \mathcal{M}) = \int_{\mathcal{M}} \left[\beta(x - \hat{x}_1)^2 + \rho\right] f_X(x)dx + \int_{\mathcal{M}_c} (x - \hat{x}_0)^2 f_X(x)dx$$

- How do we determine $\mathcal{M}$ and $\mathcal{M}_c$?

$$\beta(x - \hat{x}_1)^2 + \rho \underset{x \in \mathcal{M}}{\overset{x \in \mathcal{M}_c}{\gtrless}} (x - \hat{x}_0)^2$$

$$p(x) \triangleq (\beta - 1)x^2 + 2(\hat{x}_0 - \beta\hat{x}_1)x + \beta(\hat{x}_1)^2 - (\hat{x}_0)^2 + \rho$$



Part 4. An Iterative Algorithm: Modified Lloyd-Max

Step 0: Initialize

$$\mathcal{M}_c^{(0)} = \left[\underline{x}^{(0)}, \overline{x}^{(0)}\right]$$

Step 1: Compute representation points

$$\hat{x}_0^{(i)} = \mathsf{E}[X | X \in \mathcal{M}_c^{(i-1)}]$$

$$\hat{x}_1^{(i)} = -\frac{\Pr(X \in \mathcal{M}_c^{(i-1)})}{1 - \Pr(X \in \mathcal{M}_c^{(i-1)})}\hat{x}_0^{(i)}$$

Step 2: Update *p*(*x*)

$$p^{(i)}(x) := (\beta - 1)x^2 + 2(\hat{x}_0^{(i)} - \beta\hat{x}_1^{(i)})x + \beta(\hat{x}_1^{(i)})^2 - (\hat{x}_0^{(i)})^2 + \rho$$

Step 3: Compute roots and update M_c

$$\underline{x}^{(i)} := \min\{\texttt{roots}(p^{(i)}(x))\}$$

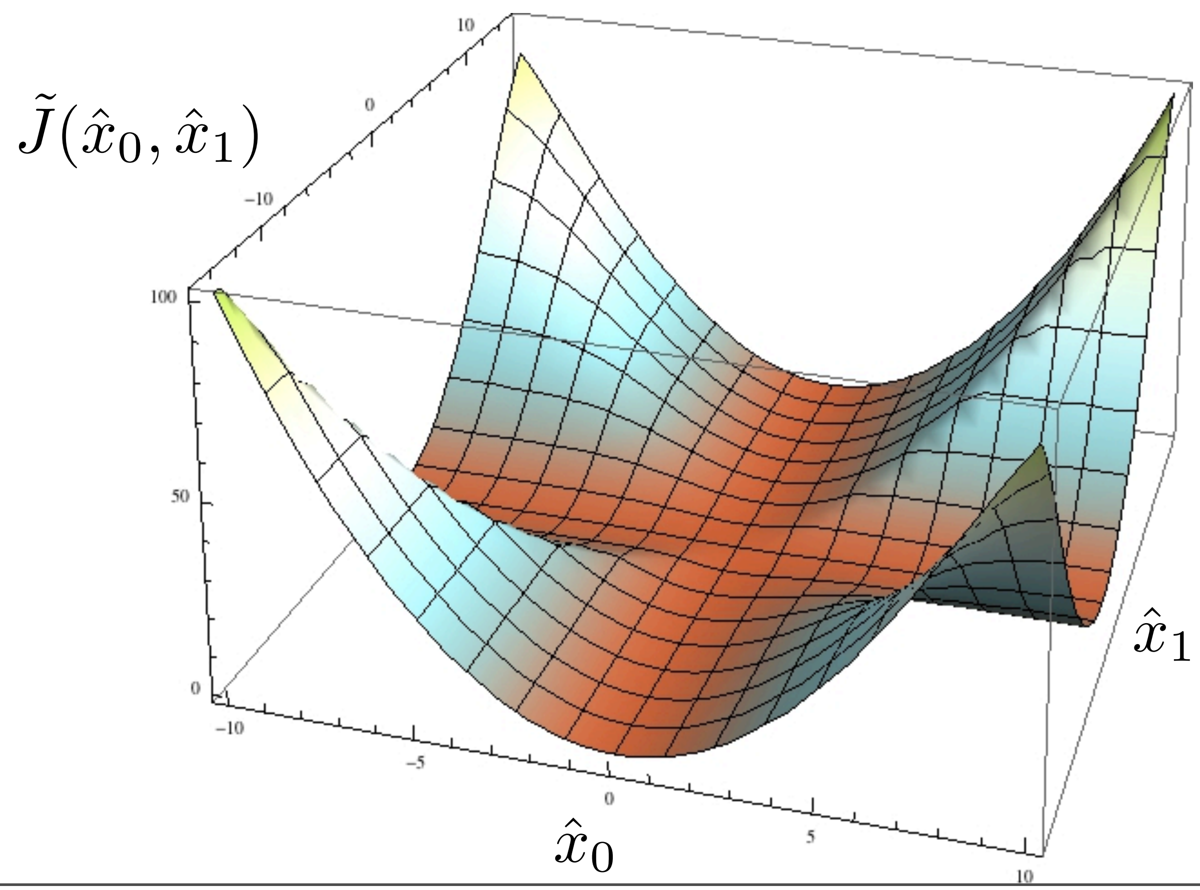
$$\overline{x}^{(i)} := \max\{\texttt{roots}(p^{(i)}(x))\}$$

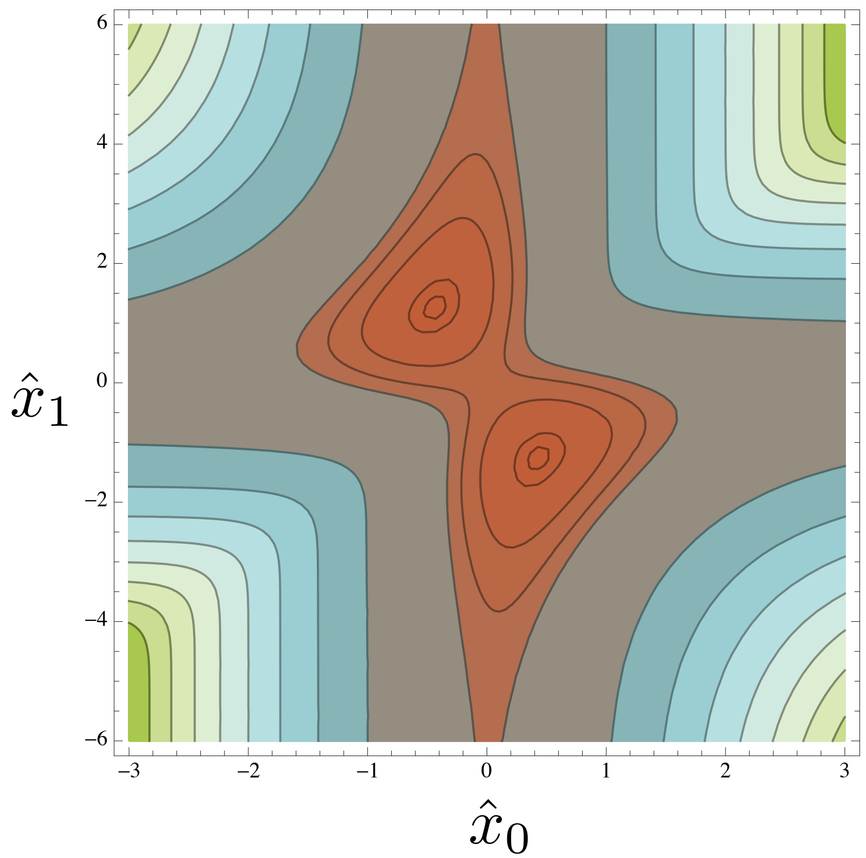
Part 4. An Iterative Algorithm: Modified Lloyd-Max

• Question: Does the MLM algorithm converge?

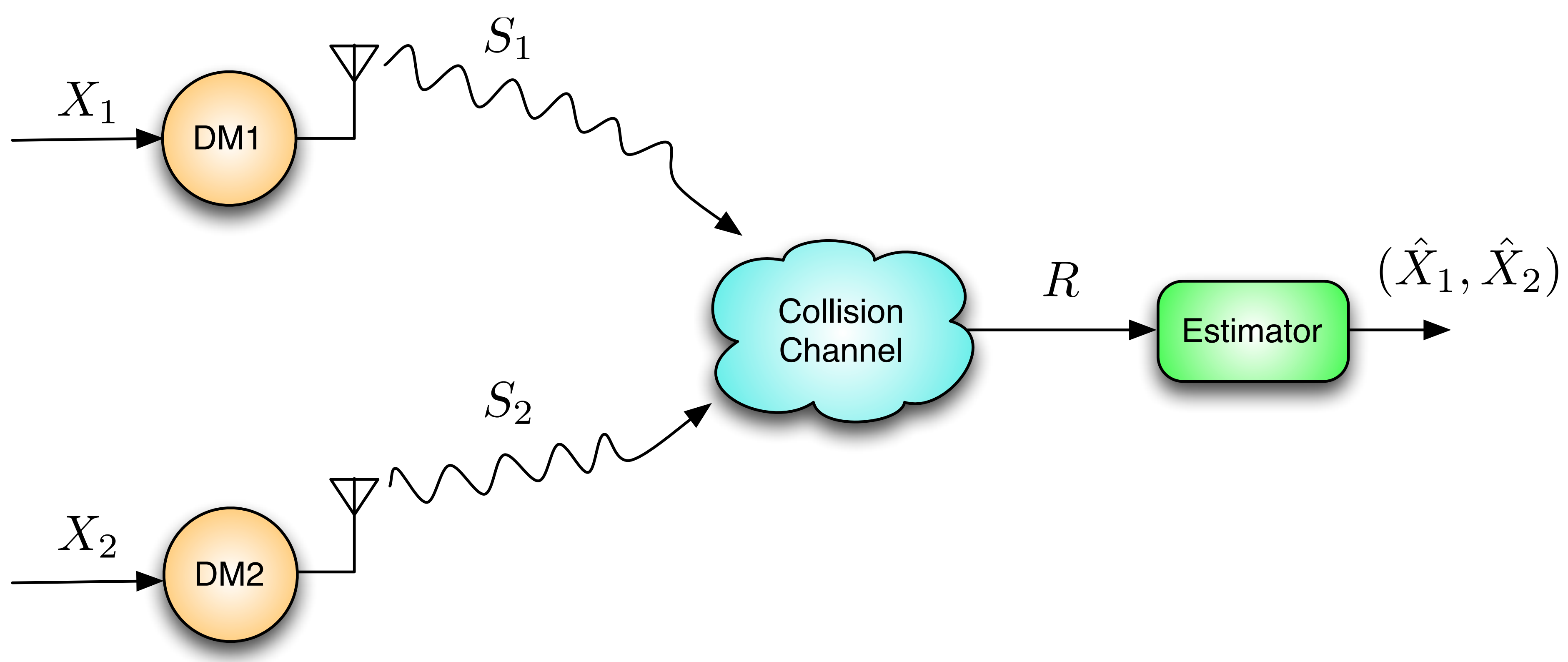
Conjecture: Provided $f_X(x) = \mathcal{N}(0, \sigma^2)$ *, the algorithm will always converge to* (0,0) or *a global minimum.*

• Evidence: MLM is a descending algorithm & cost has no local minima





A few consequences of our results



- The policy for DM*i* was reduced to a pair of real numbers $(\underline{x}_i, \overline{x}_i)$
- Reduce problem as a finite dimensional static team
- An implementable algorithm
- Ideas on how to extend to multidimensional dynamical systems

Conclusion and Final Remarks

- A new channel model for networked estimation/control
- Ran into a familiar problem along the way
- Used different set of tools: randomized policies + infinite dimensional LP
- Exploited the idea of signaling and quantization to find jointly optimal communication and estimation policies