Remote Estimation Games over Shared Networks

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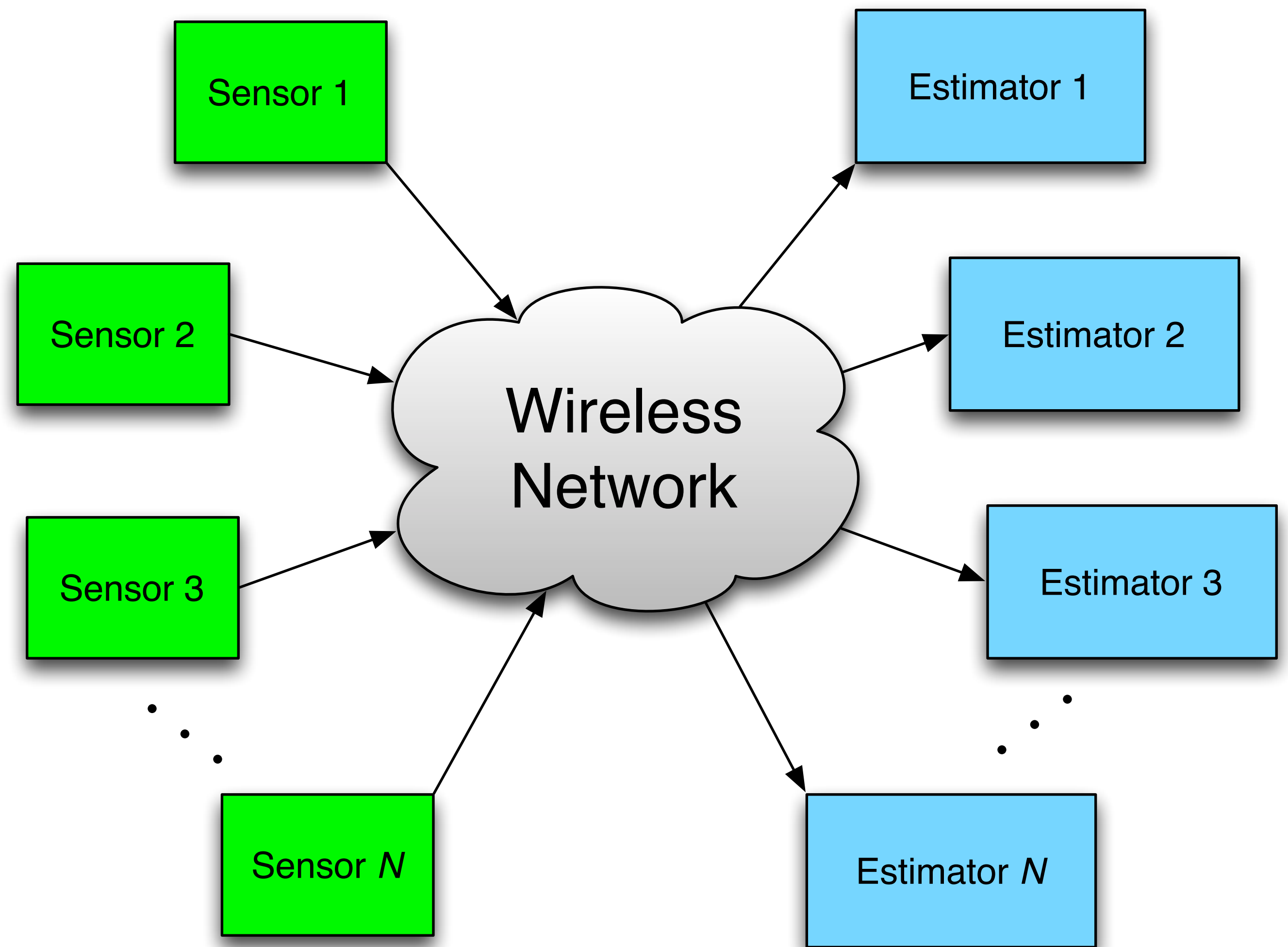
Motivation

Context

- Networked control systems
- Decision Makers cooperate or compete to achieve certain goals

Network models

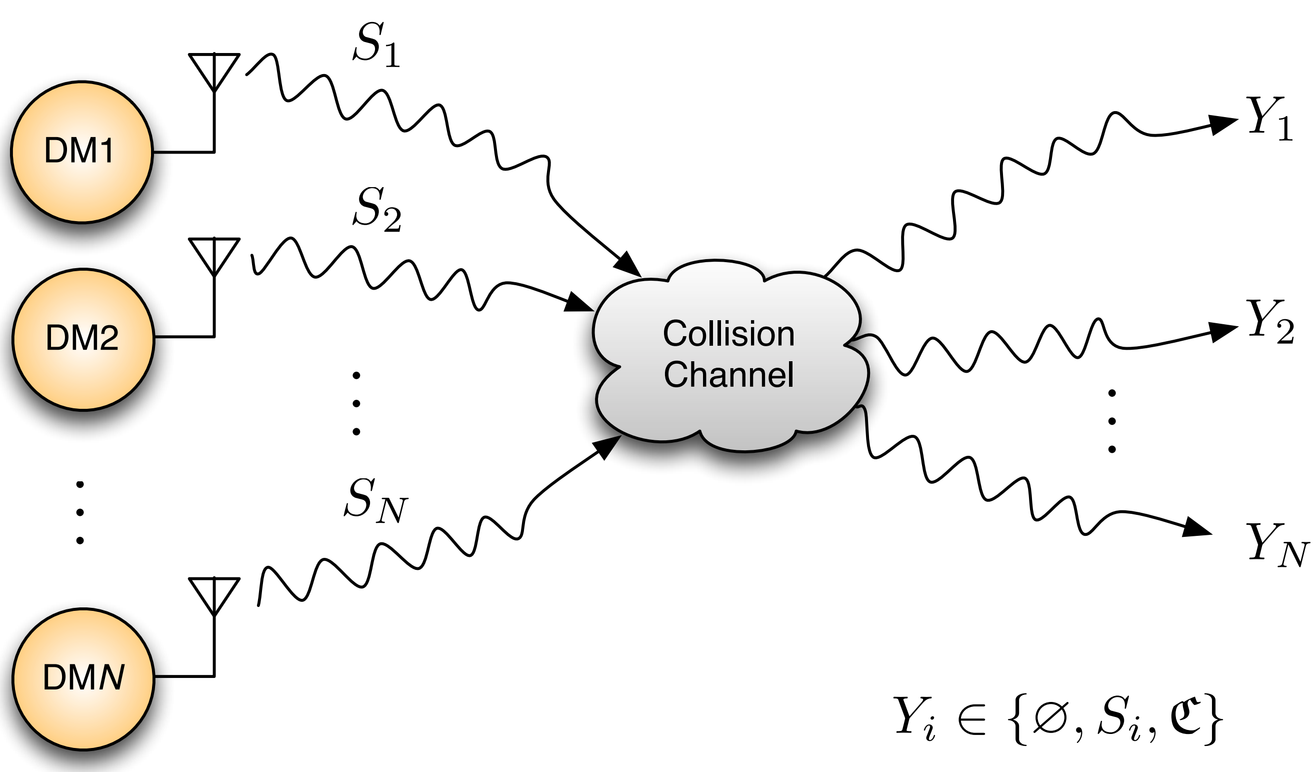
- Incomplete graphs
- Rate-limited point-to-point channels
- Additive White Gaussian Noise
- Analog Erasure channel



What about interference?

Interference

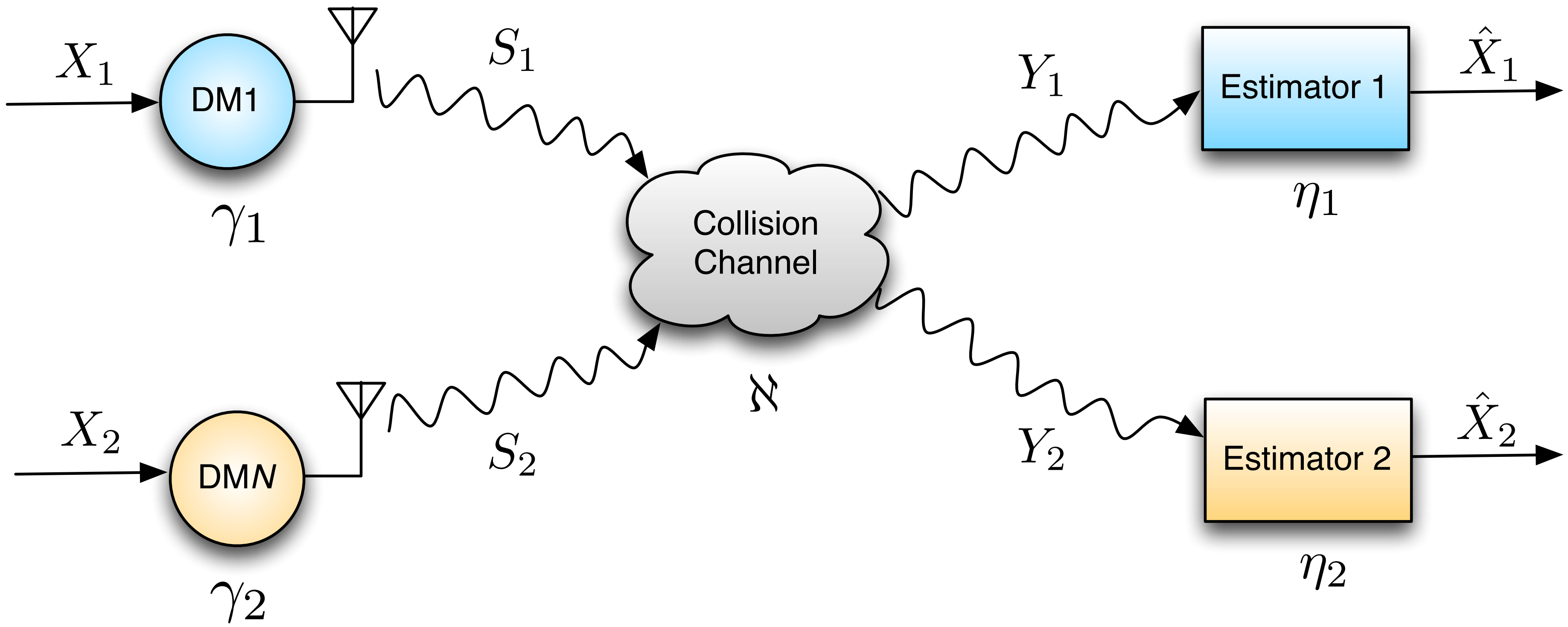
- Multiple agents sharing a communication medium
- Physical layer: Multiple Access Channel, Interference Channel
- MAC/Network layer: Collision Channel



Channel Model

- DM chooses to transmit or not
- Collision when two or more DMs transmit
- Simplest model for interference

Problem Statement



$$X_i \sim \mathcal{N}(0, \sigma_i^2)$$
$$X_1 \perp\!\!\!\perp X_2$$

$$U_i = \gamma_i(X_i) \in \{0, 1\}$$

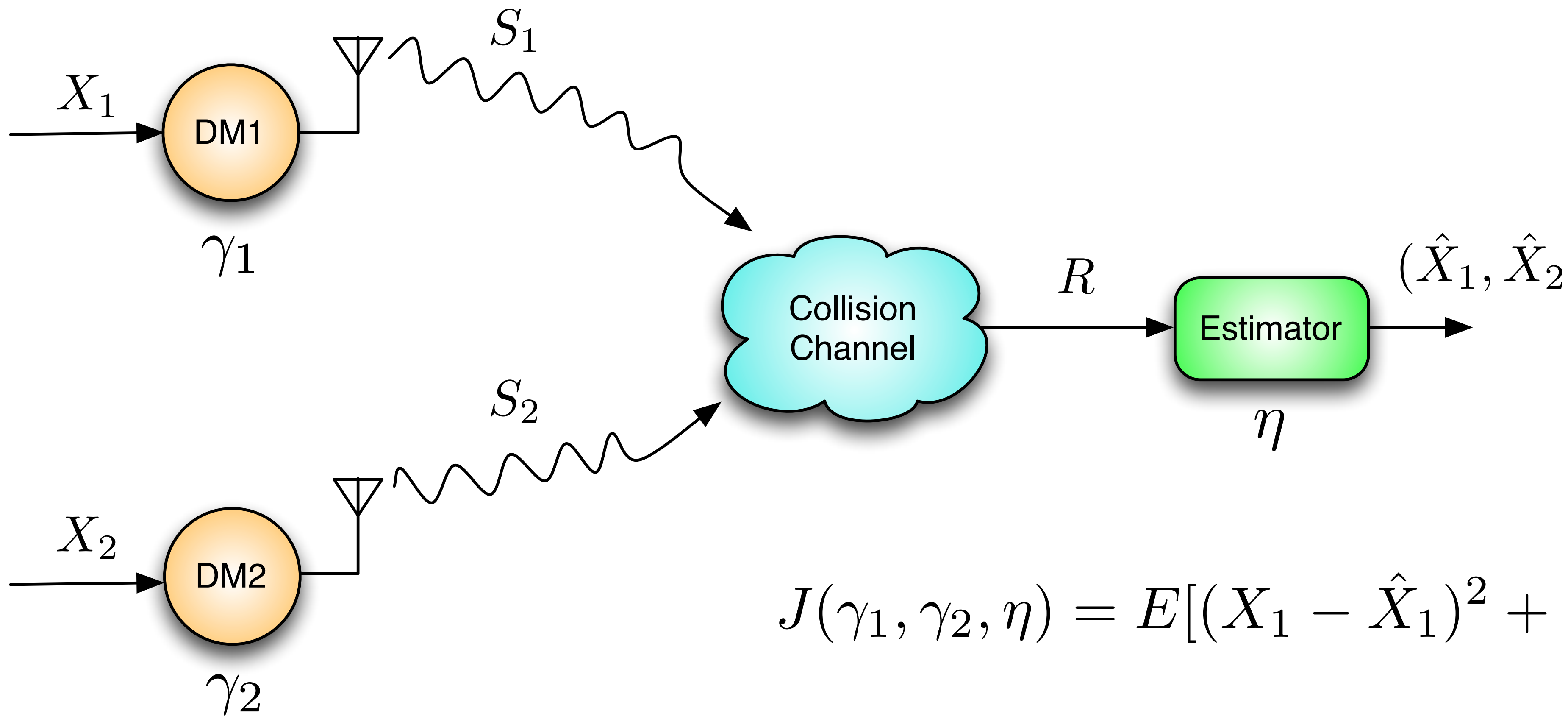
$$S_i = \begin{cases} X_i, & \text{if } U_i = 1 \\ \varnothing, & \text{if } U_i = 0 \end{cases}$$

$$\aleph(s_1, s_2) = \begin{cases} (\varnothing, \varnothing), & \text{if } s_1 = \varnothing, s_2 = \varnothing \\ (x_1, \varnothing), & \text{if } s_1 = x_1, s_2 = \varnothing \\ (\varnothing, x_2), & \text{if } s_1 = \varnothing, s_2 = x_2 \\ (\mathfrak{C}, \mathfrak{C}), & \text{if } s_1 = x_1, s_2 = x_2 \end{cases}$$

Each sensor-estimator pair minimizes its own cost functional

$$J_i(\gamma_i, \gamma_j, \eta_i) = E[(X_i - \hat{X}_i)^2]$$

Previous work



$$J(\gamma_1, \gamma_2, \eta) = E[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2]$$

Vasconcelos & Martins (Allerton '13)

- Team decision problem - Focus on full cooperation
- Proved the optimality of threshold policies (asymmetric in general)

Our main results

Focus on competitive behavior

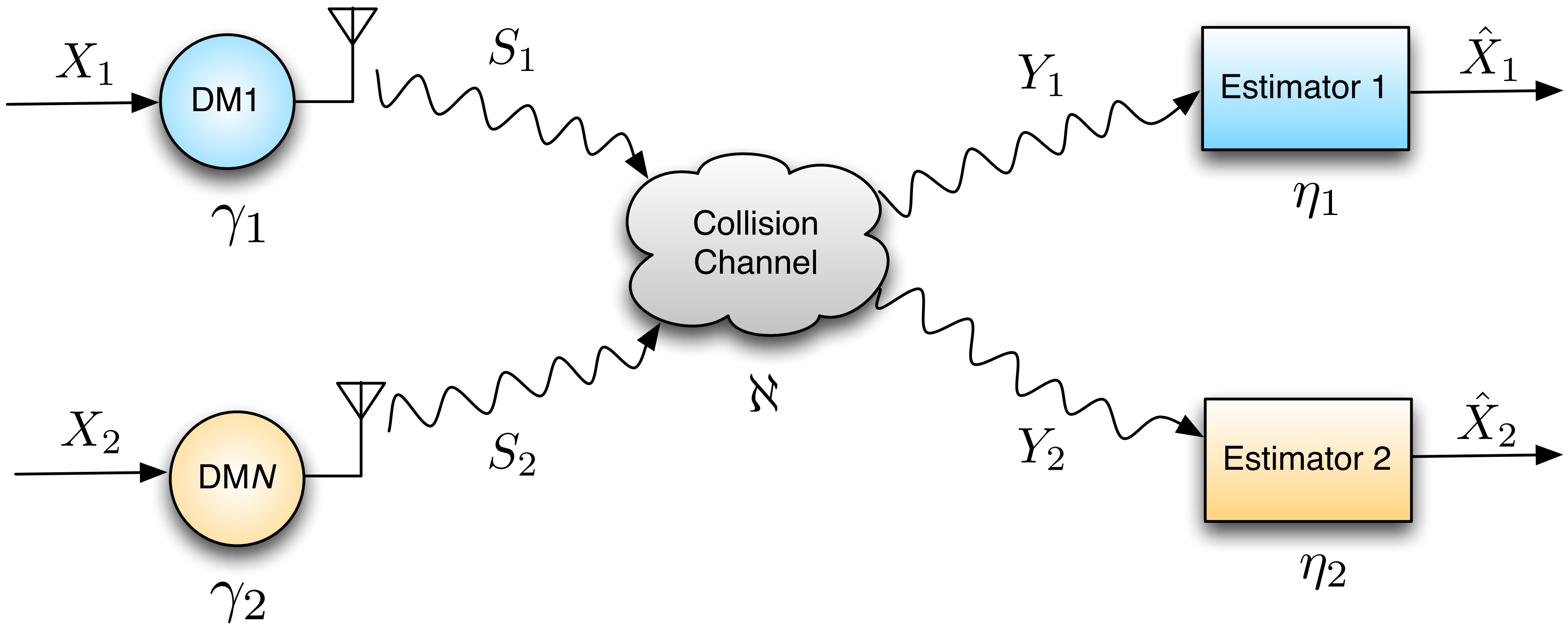
1. Obtain the structure of security and Nash equilibrium policies

   collision channel without and with capture

2. Establish a connection with optimal quantization theory

3. Policy design using the Lloyd-Max algorithm

Part I. The collision channel without capture



Assume DM2 transmits with prob. 1 - selfish behavior

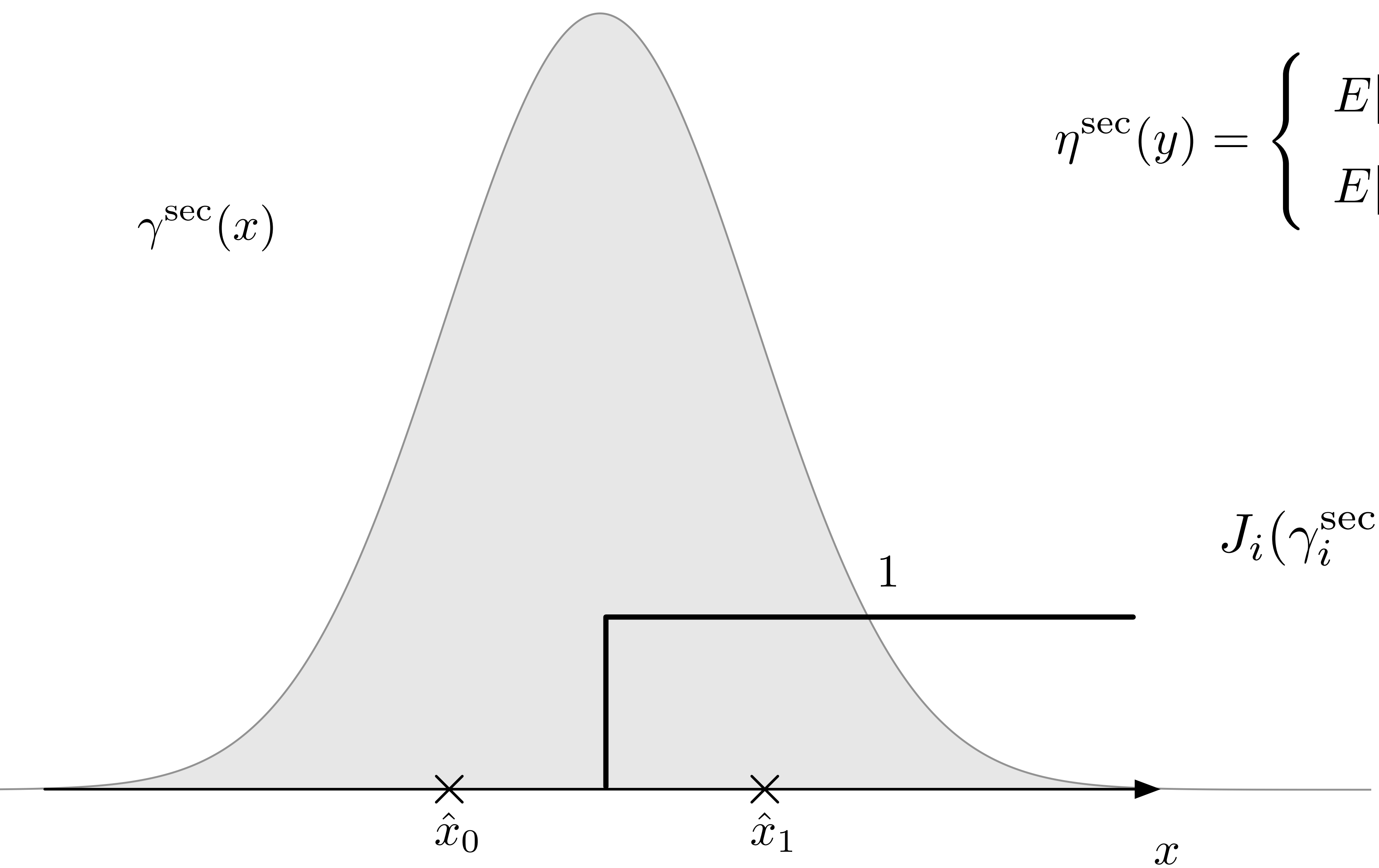
Worst case scenario for DM1

Estimator 1 only receives $Y_1 \in \{\varnothing, \mathfrak{C}\}$

Security policies

When the channel is always occupied by the opponent:

Best communication policy



Best estimation policy

$$\eta^{\text{sec}}(y) = \begin{cases} E[X|X \geq 0] = \sqrt{\frac{2}{\pi}}\sigma, & \text{if } y = \mathfrak{C} \\ E[X|X < 0] = -\sqrt{\frac{2}{\pi}}\sigma, & \text{if } y = \varnothing. \end{cases}$$

Cost

$$J_i(\gamma_i^{\text{sec}}, \gamma_j^{\text{self}}, \eta_i^{\text{sec}}) = \left(1 - \frac{2}{\pi}\right)\sigma_i^2$$

The security policy is determined by the optimal 1 bit quantizer

Security policies

Proposition 1:
A security policy for DM*i* in the game over the collision channel has a single threshold structure of the form

$$\gamma_i^{\mathrm{sec}}(x_i) = \begin{cases} 1, & x_i \geq 0; \\ 0, & x_i < 0. \end{cases}$$

If both DMs use security policies, their incurred costs are:

$$J_i^{\mathrm{sec}} = \frac{3}{4}\left(1 - \frac{2}{\pi}\right)\sigma_i^2$$

Security policies

Example:

$$\sigma_1^2 = 1 \qquad J_1(\gamma_1^{\text{sec}}, \gamma_2^{\text{self}}, \eta_1^{\text{sec}}) = 0.3634 \qquad J_1^{\text{sec}} = \frac{3}{4}\left(1 - \frac{2}{\pi}\right)\sigma_1^2 = \underline{0.2725}$$

$$\sigma_2^2 = 2 \qquad J_2(\gamma_2^{\text{self}}, \gamma_1^{\text{sec}}, \eta_2^{\text{self}}) = 1 \qquad J_2^{\text{sec}} = \frac{3}{4}\left(1 - \frac{2}{\pi}\right)\sigma_2^2 = \underline{0.5450}$$

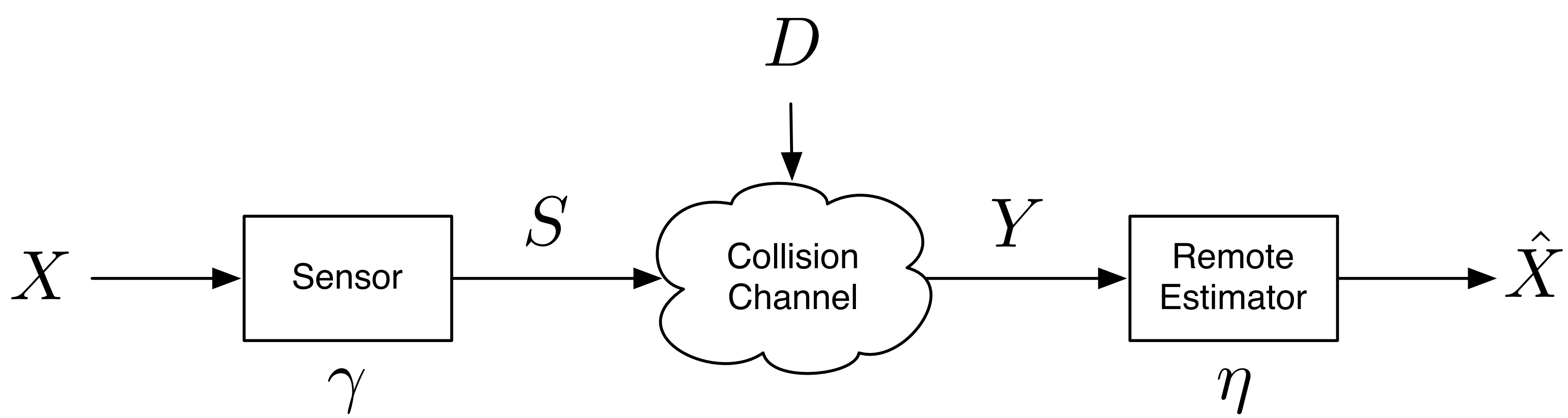
A security policy accesses the channel with probability $\beta = 0.5$

Question:
What is the structure of the optimal communication policy when the channel is occupied with probability $\beta < 1$?

Structure of Nash equilibrium policies

Analysis from the perspective of a single DM

Assume the opponent transmits with probability β



$$D \sim \mathcal{B}(\beta)$$

$$U = \gamma(X) \in \{0,1\}$$

$$U = 0 \Rightarrow Y = \varnothing$$

$$U = 1, D = 0 \Rightarrow Y = X$$

$$U = 1, D = 1 \Rightarrow Y = \mathfrak{C}$$

$$\eta(X) = X$$

$$\eta(\varnothing) = \hat{x}_0$$

$$\eta(\mathfrak{C}) = \hat{x}_1$$

$$J(\gamma,\eta) = E[(X-\hat{x}_0)^2|U=0]\Pr(U=0) + E[\beta(X-\hat{x}_1)^2|U=1]\Pr(U=1)$$

Binary quantization with asymmetric distortion

Structure of Nash equilibrium policies

$$J(\mathbb{A}_0, \hat{x}_0, \hat{x}_1) = \int_{\mathbb{A}_0} (x - \hat{x}_0)^2 f_X(x) dx + \int_{\mathbb{R} \setminus \mathbb{A}_0} \beta (x - \hat{x}_1)^2 f_X(x) dx$$

Necessary optimality condition: $x \in \mathbb{A}_0^* \Leftrightarrow (x - \hat{x}_0)^2 \leq \beta (x - \hat{x}_1)^2$

$$p(x) \stackrel{\text{def}}{=} (x - \hat{x}_0)^2 - \beta (x - \hat{x}_1)^2$$

$$\mathbb{A}_0^* = \{x \in \mathbb{R} \mid p(x) \leq 0\}$$

$$p''(x) \geq 0 \qquad \Rightarrow \qquad \mathbb{A}_0^* \text{ is a convex set}$$

Theorem 1:
The Nash equilibrium policies for the game over the collision channel without capture have the following threshold structure

$$\gamma^{\text{nash}}(x) = \begin{cases} 0, & \text{if } \tau_1 \leq x \leq \tau_2; \\ 1, & \text{otherwise.} \end{cases}$$

Design via Lloyd-Max Algorithm

1. From a pair of representation points compute the roots of $p(x)$

$$\hat{x}^{(k)} = (\hat{x}_0^{(k)}, \hat{x}_1^{(k)})$$

$$\tau_1(\hat{x}^{(k)}) = \frac{\hat{x}_0^{(k)} + \sqrt{\beta}\hat{x}_1^{(k)}}{1+\sqrt{\beta}}$$

$$\tau_2(\hat{x}^{(k)}) = \frac{\hat{x}_0^{(k)} - \sqrt{\beta}\hat{x}_1^{(k)}}{1-\sqrt{\beta}}$$

2. The new representation points are the centroids of $\mathbb{A}_0^{(k)}, \mathbb{A}_1^{(k)} = \mathbb{R} \backslash \mathbb{A}_0^{(k)}$

$$\mathbb{A}_0^{(k)} = \left[\tau_1(\hat{x}^{(k)}), \tau_2(\hat{x}^{(k)})\right]$$

$$\hat{x}^{(k+1)} = \begin{bmatrix} \frac{1}{\int_{\mathbb{A}_0^{(k)}} f_X(x)dx} \\ \frac{-1}{1-\int_{\mathbb{A}_0^{(k)}} f_X(x)dx} \end{bmatrix} \int_{\mathbb{A}_0^{(k)}} x f_X(x)dx$$

This algorithm converges globally to a local minimum

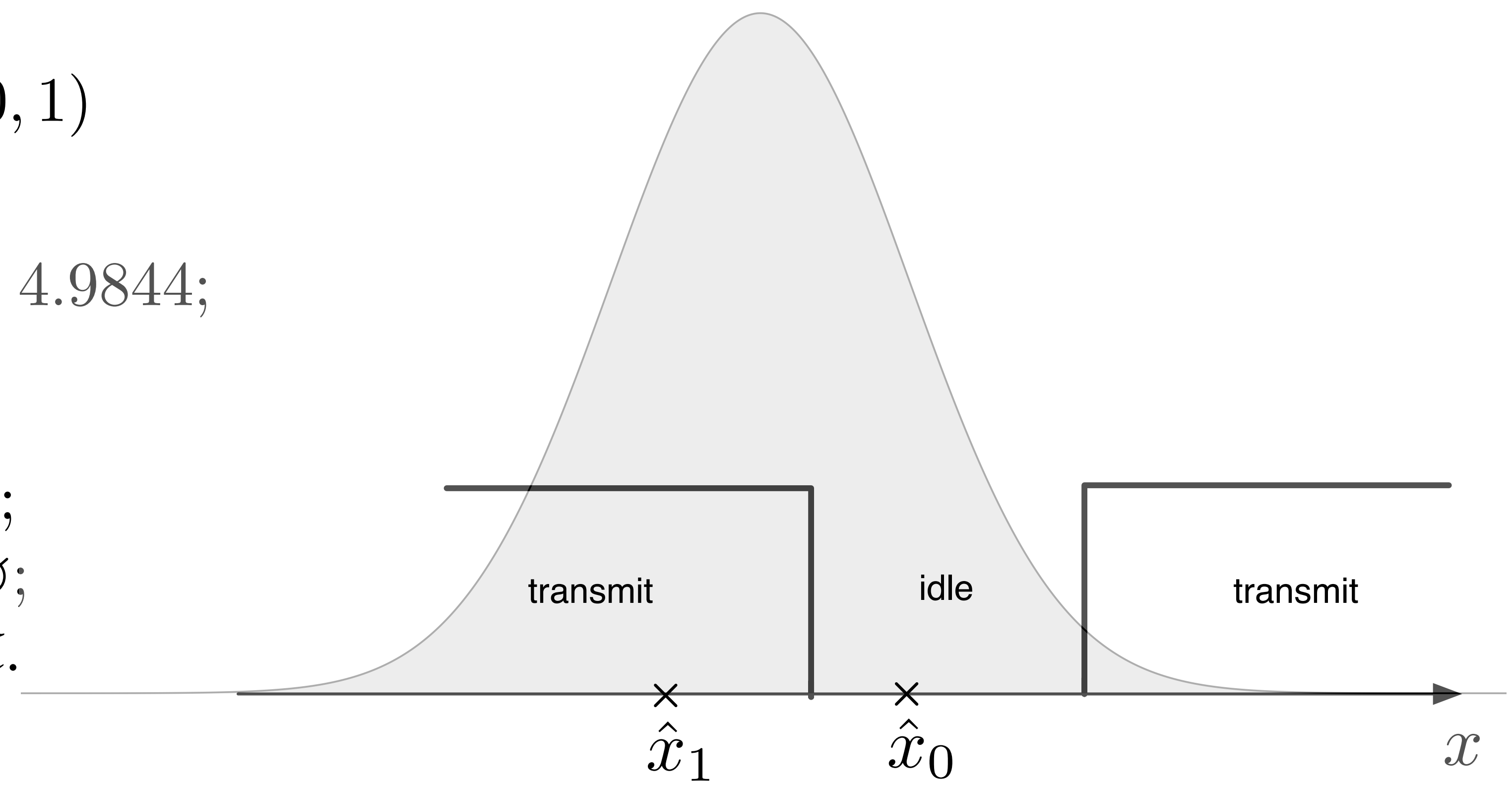
Design via Lloyd-Max Algorithm

Example 1: $\beta = 0.5, X \sim \mathcal{N}(0,1)$

$$\gamma^*(x) = \begin{cases} 0, & \text{if } 0.3813 \leq x \leq 4.9844; \\ 1, & \text{otherwise.} \end{cases}$$

$$\eta^*(y) = \begin{cases} x, & \text{if } y = x; \\ +1.0554, & \text{if } y = \varnothing; \\ -0.5720, & \text{if } x = \mathfrak{C}. \end{cases}$$

$$J^* = 0.2488$$



Example 2: $X_1 \sim \mathcal{N}(0,1) \quad X_2 \sim \mathcal{N}(0,2)$

$$\gamma_1^{\text{nash}}(x_1) = \begin{cases} 0, & +0.2736 \leq x_1 \leq +6.6828; \\ 1, & \text{otherwise} \end{cases}$$

$$\gamma_2^{\text{nash}}(x_2) = \begin{cases} 0, & +0.3869 \leq x_2 \leq +9.4510; \\ 1, & \text{otherwise} \end{cases}$$

$$J_1^{\text{nash}} = 0.2786$$

$$J_2^{\text{nash}} = 0.5573$$

Remarks

1. The structural result is independent of the densities of X_1, X_2

2. The convergence of the Lloyd-Max algorithm depends on the pdfs

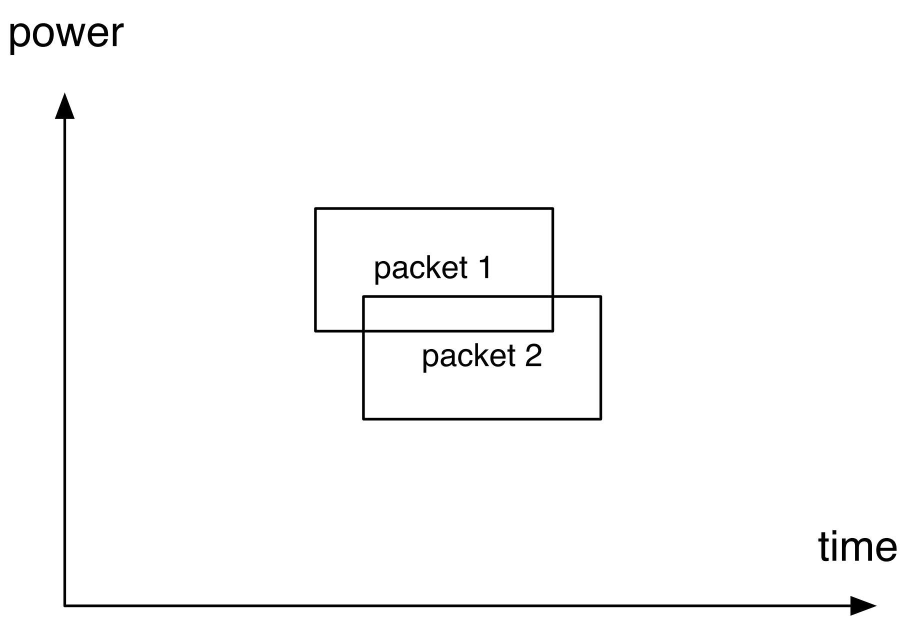
3. The Nash equilibrium policies perform worse than the security ones:

$$J_1^{\text{nash}} = 0.2786 \qquad J_1^{\text{sec}} = 0.2725$$

$$J_2^{\text{nash}} = 0.5573 \qquad J_2^{\text{sec}} = 0.5450$$

There is an incentive to be conservative even in the absence of communication costs

Part II. Collision channel with capture



Capture mechanism:

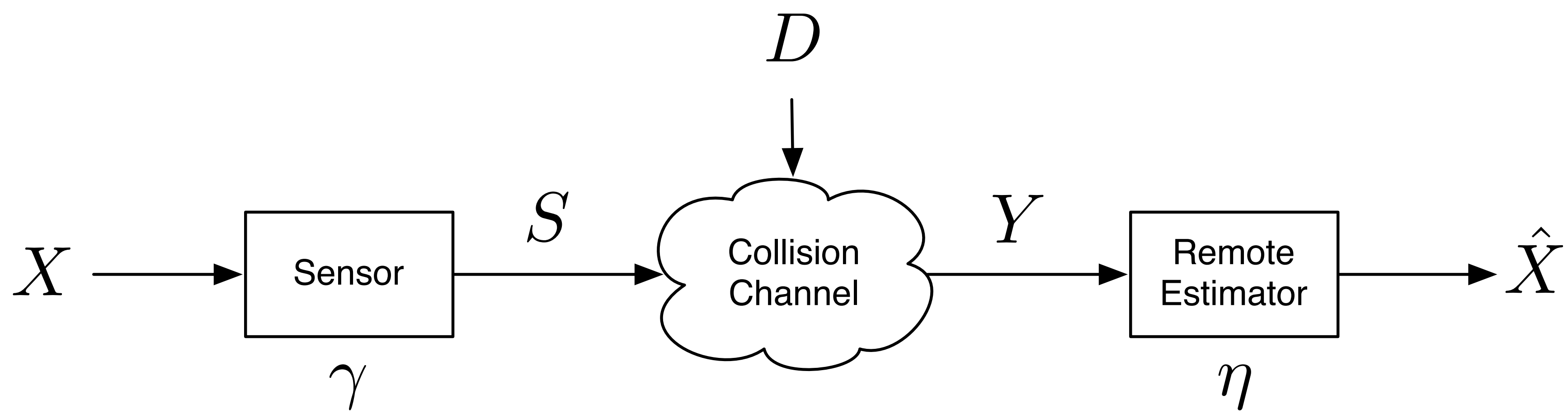
In case of a collision, the packet transmitted with the highest power captures the channel and the other is lost.

Allow DMs to choose among 3 power levels: $U_i \in \{0, 1, 2\}$

Cost functional must take into account the communication cost:

$$J_i^{\text{cap}}(\gamma_i, \gamma_j, \eta_i) = E[(X_i - \hat{X}_i)^2 + \rho U_i]$$

Part II. Collision channel with capture



$$\Pr(D = i) = \beta_i, \;\; i \in \{0, 1, 2\}$$

$$U = 0 \Rightarrow Y = \varnothing \qquad \eta(X) = X$$

$$U > 0, D < U \Rightarrow Y = X \qquad \eta(\varnothing) = \hat{x}_0$$

$$U = 1, D \geq 1 \Rightarrow Y = \mathfrak{C}_1 \qquad \eta(\mathfrak{C}_1) = \hat{x}_1$$

$$U = 2, D = 2 \Rightarrow Y = \mathfrak{C}_2 \qquad \eta(\mathfrak{C}_2) = \hat{x}_2$$

$$\begin{aligned} J^{\text{cap}}(\gamma, \eta) = & E[(X - \hat{x}_0)^2 | U = 0] \Pr(U = 0) + \\ & E[(\beta_1 + \beta_2)(X - \hat{x}_1)^2 + \rho | U = 1] \Pr(U = 1) + \\ & E[\beta_2 (X - \hat{x}_2)^2 + 2\rho | U = 2] \Pr(U = 2) \end{aligned}$$

Ternary quantization problem with asymmetric distortion

Security Policies

Worst case scenario: the opponent always transmits with full power

$$\beta_2 = 1$$

$$J^{\text{cap}} = \int_{\mathbb{A}_0} (x - \hat{x}_0)^2 f_X(x)dx + \int_{\mathbb{A}_1} [(x - \hat{x}_1)^2 + \rho] f_X(x)dx + \int_{\mathbb{A}_2} [(x - \hat{x}_2)^2 + 2\rho] f_X(x)dx$$

Necessary optimality conditions:

$$x \in \mathbb{A}_0^* \Leftrightarrow h_1(x) \leq 0, h_2(x) \leq 0$$
$$x \in \mathbb{A}_1^* \Leftrightarrow h_1(x) > 0, h_3(x) \leq 0$$
$$x \in \mathbb{A}_2^* \Leftrightarrow h_2(x) > 0, h_3(x) > 0$$

$$h_1(x) = 2x(\hat{x}_1 - \hat{x}_0) - (\hat{x}_1^2 - \hat{x}_0^2 + \rho)$$
$$h_2(x) = 2x(\hat{x}_2 - \hat{x}_0) - (\hat{x}_2^2 - \hat{x}_0^2 + 2\rho)$$
$$h_3(x) = 2x(\hat{x}_2 - \hat{x}_1) - (\hat{x}_2^2 - \hat{x}_1^2 + \rho)$$

$\mathbb{A}_i^*, \ i \in \{0, 1, 2\}$ are convex

Theorem 2:
The security policy for the game over the collision channel with capture is determined by a regular quantizer (convex quantization regions).

Nash equilibrium policies

$$J^{\text{cap}} = \int_{\mathbb{A}_0} (x-\hat{x}_0)^2 f_X(x)dx + \int_{\mathbb{A}_1} [(\beta_1+\beta_2)(x-\hat{x}_1)^2+\rho] f_X(x)dx + \int_{\mathbb{A}_2} [\beta_2(x-\hat{x}_1)^2+2\rho] f_X(x)dx$$

Necessary optimality conditions:

$$x \in \mathbb{A}_0^* \Leftrightarrow p_1(x) \leq 0,\ p_2(x) \leq 0$$
$$x \in \mathbb{A}_1^* \Leftrightarrow p_1(x) > 0,\ p_3(x) \leq 0$$

$$\begin{aligned} p_1(x) =& (x-\hat{x}_0)^2 - (\beta_1+\beta_2)(x-\hat{x}_1)^2 - \rho \\ p_2(x) =& (x-\hat{x}_0)^2 - \beta_2(x-\hat{x}_2)^2 - 2\rho \\ p_3(x) =& (\beta_2+\beta_1)(x-\hat{x}_2)^2 + \rho - \beta_2(x-\hat{x}_2)^2 - 2\rho \end{aligned}$$

$$p_i''(x) \geq 0 \Rightarrow \{x \in \mathbb{R} \mid p_i(x) \leq 0\} \text{ is a convex set}$$

$\mathbb{A}_0^*$ is the intersection of two convex sets $\Rightarrow \mathbb{A}_0^*$ is convex

$\mathbb{A}_1^*$ is the set difference of two convex sets $\Rightarrow \mathbb{A}_1^*$ is the union of at most two convex sets

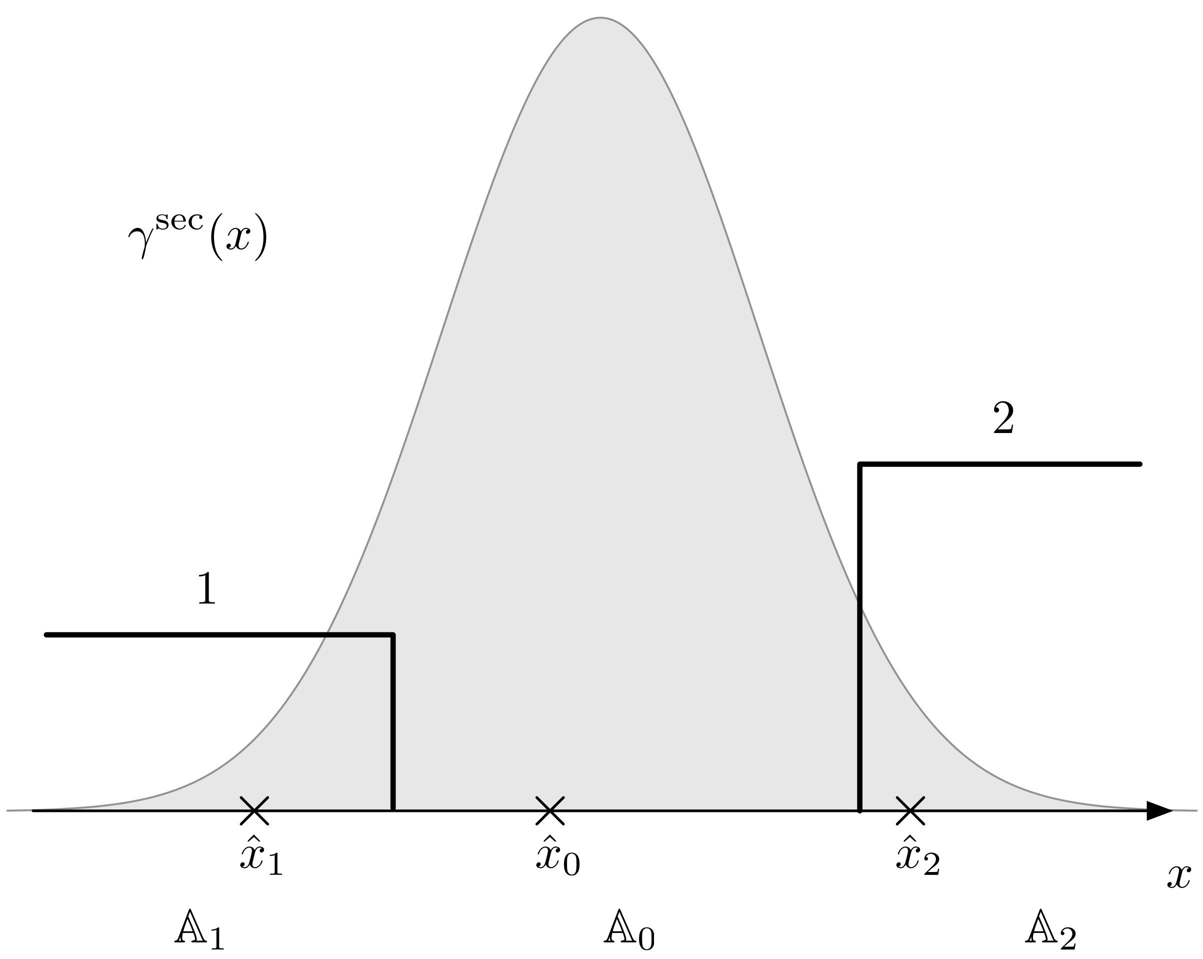
Theorem 3:
The structure of a Nash equilibrium policy for the game over the collision channel with capture is

$$\gamma^{\text{nash}}(x) = \begin{cases} 0, & \text{if } \tau_1 \leq x \leq \tau_2 \\ 1, & \text{if } \tau_3 \leq x \leq \tau_4 \text{ or } \tau_5 \leq x \leq \tau_6 \\ 2, & \text{otherwise.} \end{cases}$$

Examples

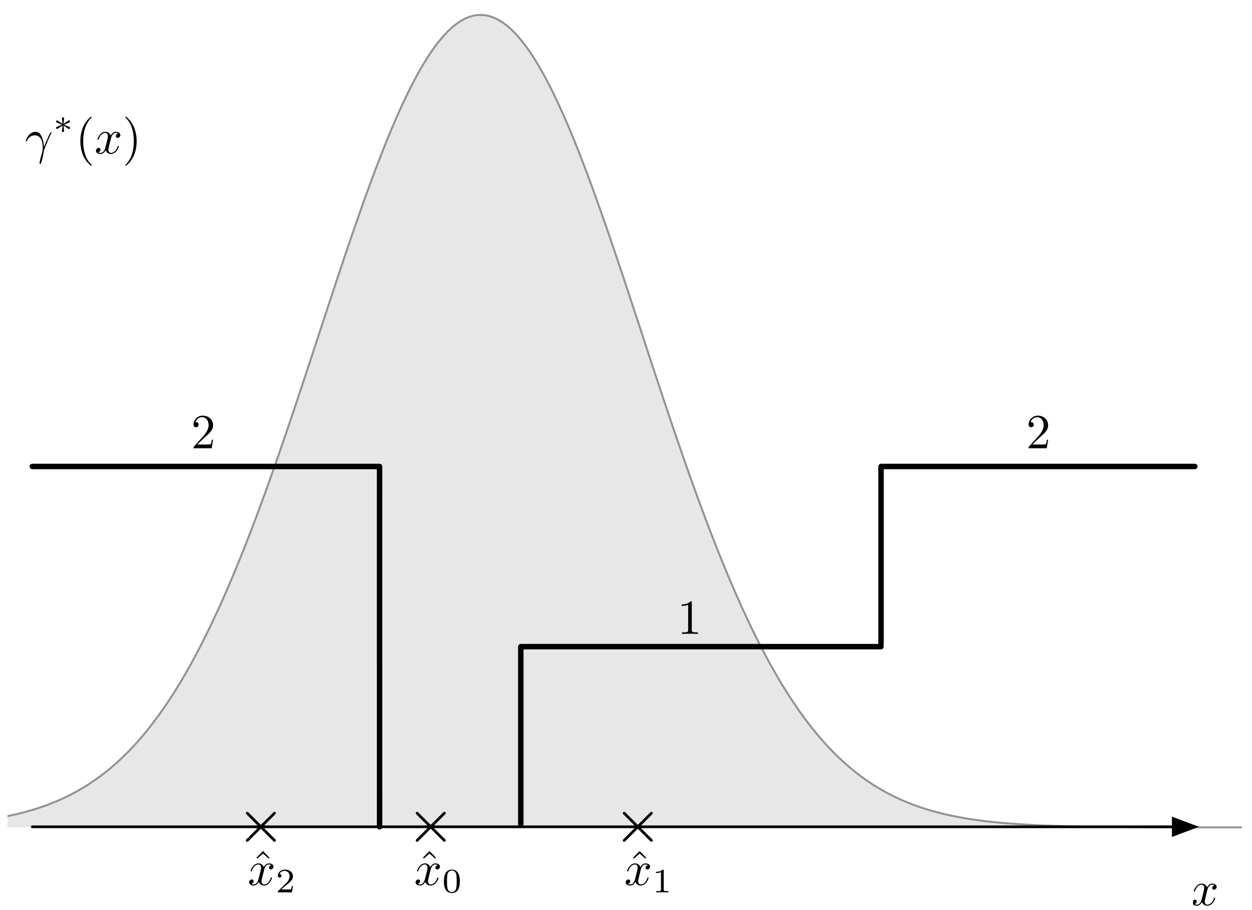
Example 1:

$$X \sim \mathcal{N}(0,1), \beta_2 = 1$$



Example 2:

$$X \sim \mathcal{N}(0,1), \beta_1 = 0.25, \beta_2 = 0.125$$



Conclusion

- Two new problems in networked estimation/control:
  - Collision channel without capture - absence of communication costs
  - Collision channel with capture - presence of communication costs
- Obtained the structure of security and Nash equilibria policies
- Results rely on optimal quantization theory with asymmetric distortion
- Several open problems:
  - Existence of optimal quantizers
  - Uniqueness of Nash equilibrium policies
  - Convergence of the Lloyd-Max algorithm
  - Dynamic games and many more!