Optimal threshold strategies for estimation over the collision channel with a communication cost

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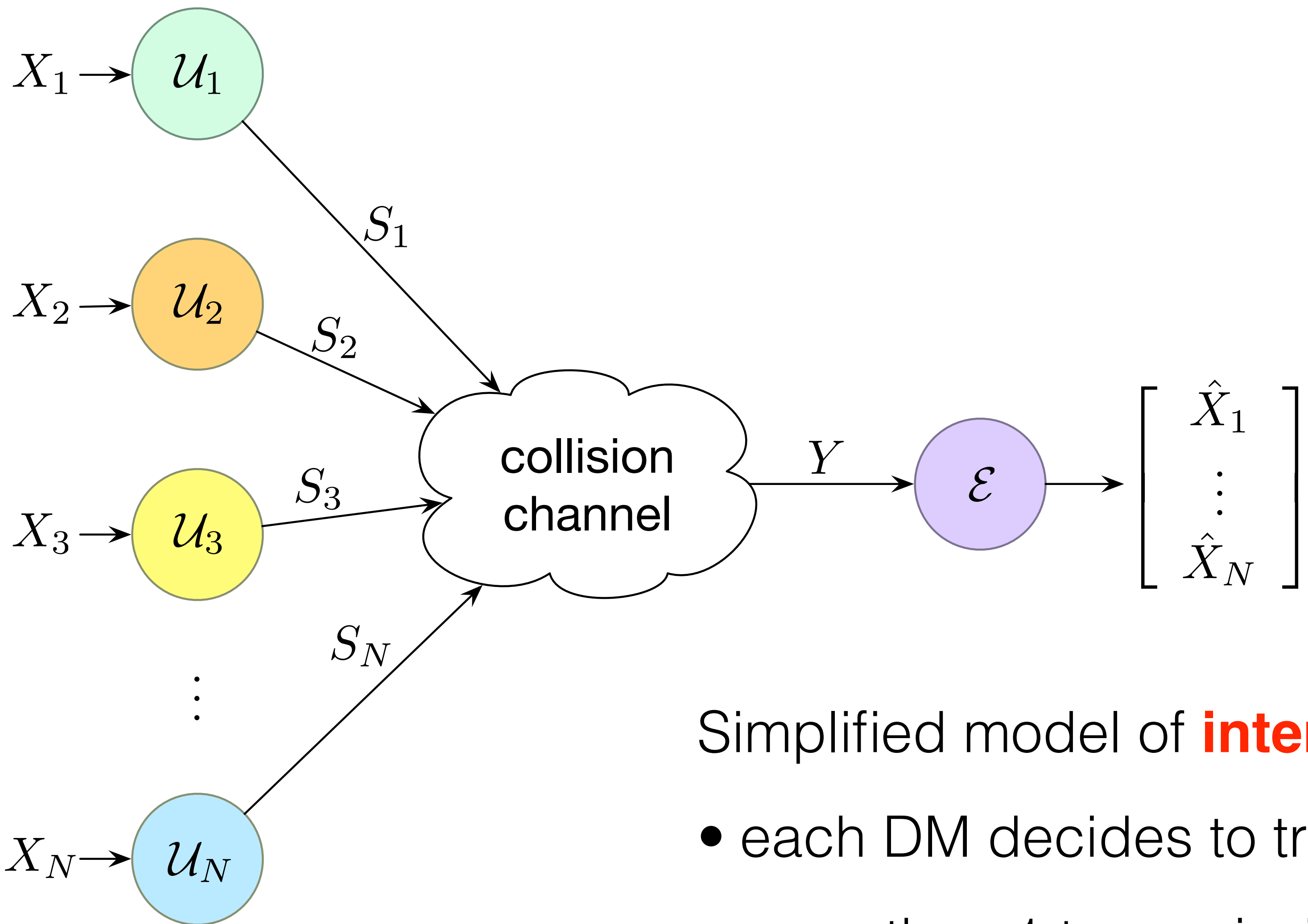
Outline

- Context and motivation:
    1. Collision channel model
    2. Previous work
- Problem formulation
- Our main results
    1. Existence of solutions
    2. Symmetry of optimal thresholds
    3. Convergence of a Lloyd-Max type algorithm
- Future work and open problems

The collision channel

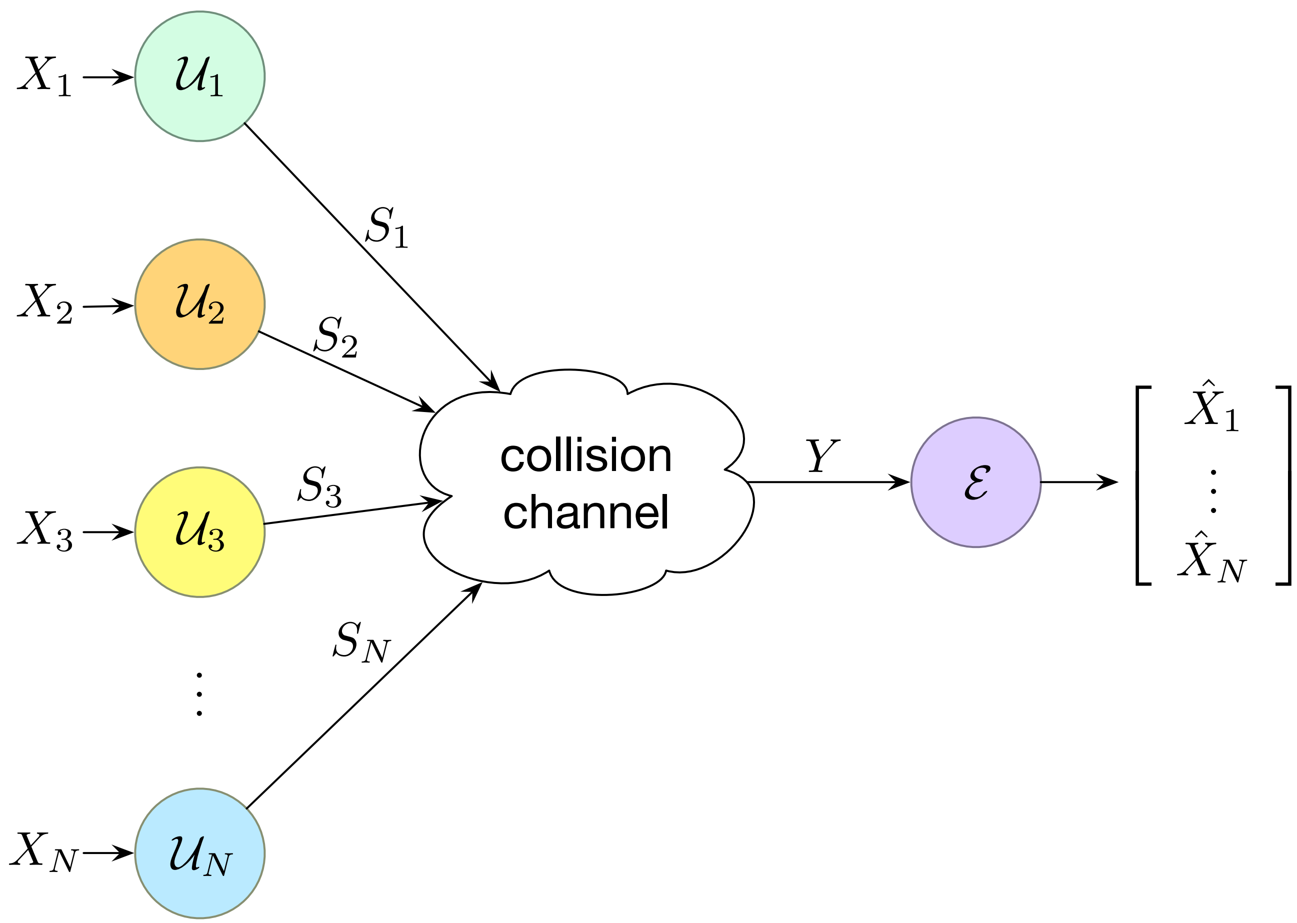


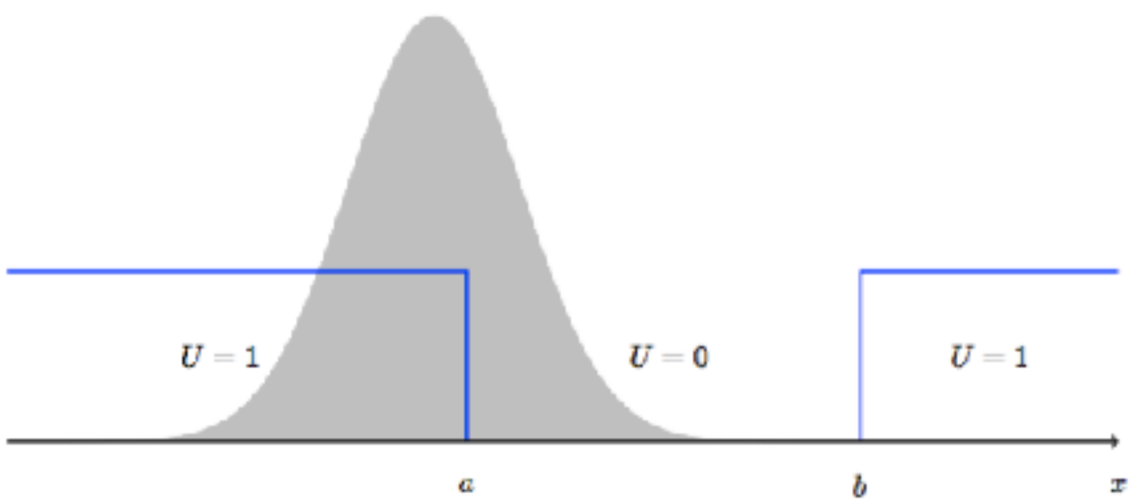
Simplified model of **interference**:

- each DM decides to transmit or not
- more than 1 transmission results in a **collision**
- no transmissions results in an **erasure**

$$\mathcal{J}(\mathcal{U}_1, \ldots, \mathcal{U}_N, \mathcal{E}) = \mathbb{E}\left[\sum_{i=1}^{N} (X_i - \hat{X}_i)^2\right]$$

Our previous work on the collision channel





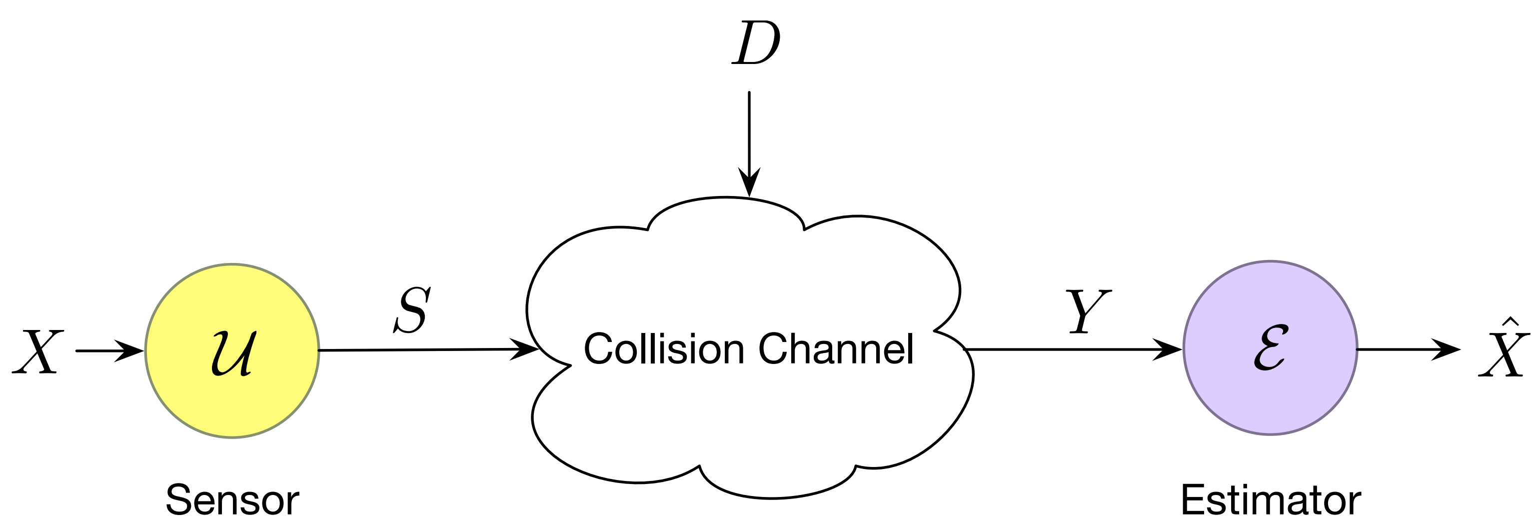
threshold policy

$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

1. Optimal policies have a **deterministic threshold structure**
2. Structural result **holds for any distribution**

[Vasconcelos and Martins - Allerton ’13]

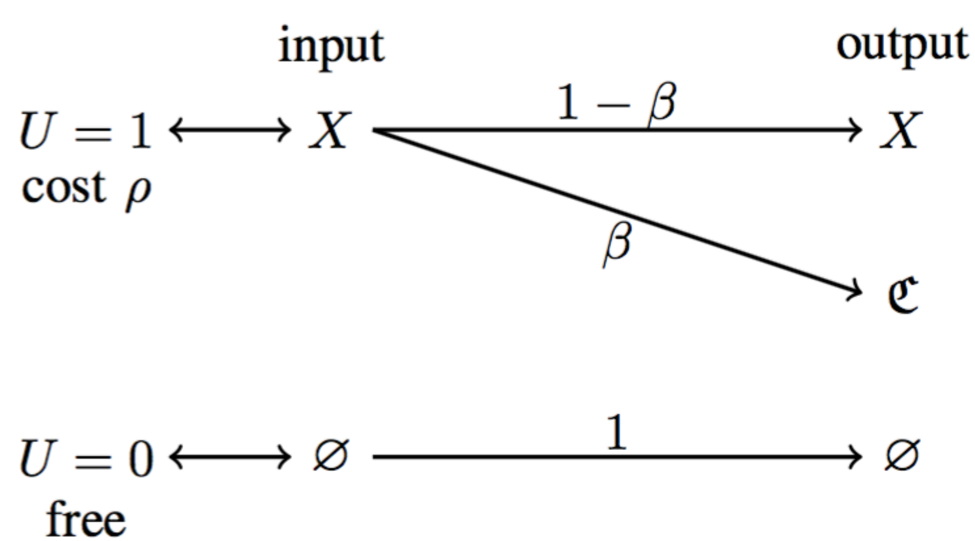
Problem formulation



$X \sim \mathcal{N}(0, \sigma^2)$

$D \sim \mathcal{B}(\beta)$

$X \perp\!\!\!\perp D$

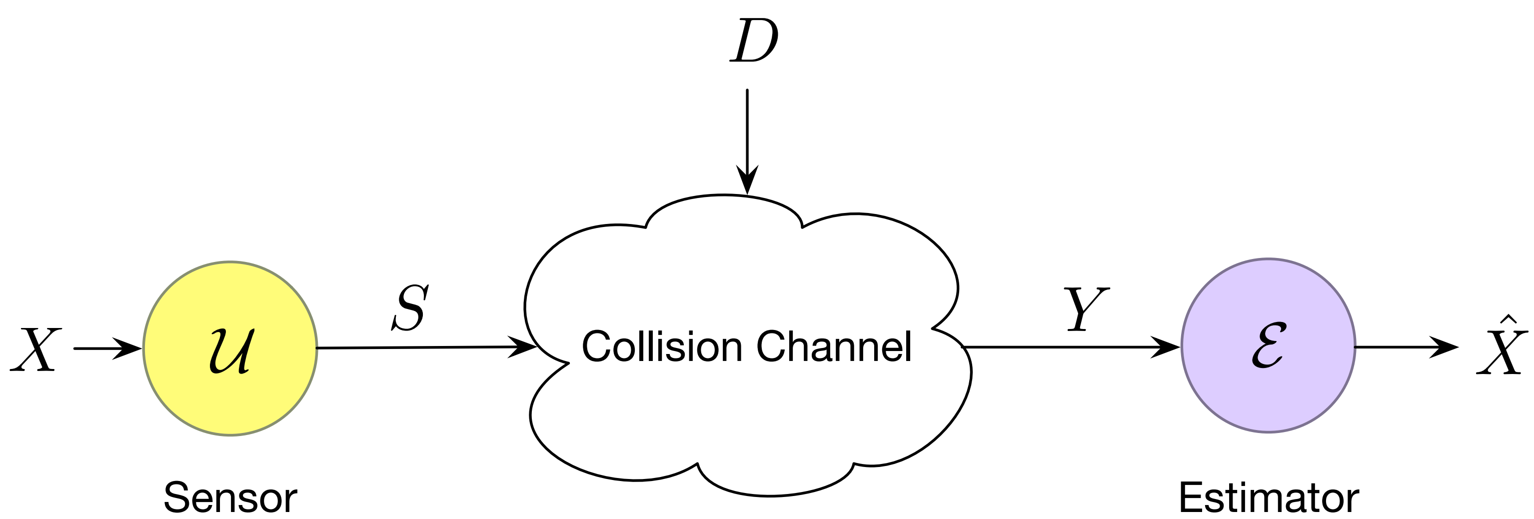


Collision channel

$$\mathcal{J}(\mathcal{U}, \mathcal{E}) = \mathbb{E}[(X - \hat{X})^2] + \rho \Pr(U = 1)$$

communication cost

Problem formulation



$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{E}(y) = \begin{cases} x & y = x \\ \hat{x}_{\varnothing} & y = \varnothing \\ \hat{x}_{\mathfrak{C}} & y = \mathfrak{C} \end{cases}$$

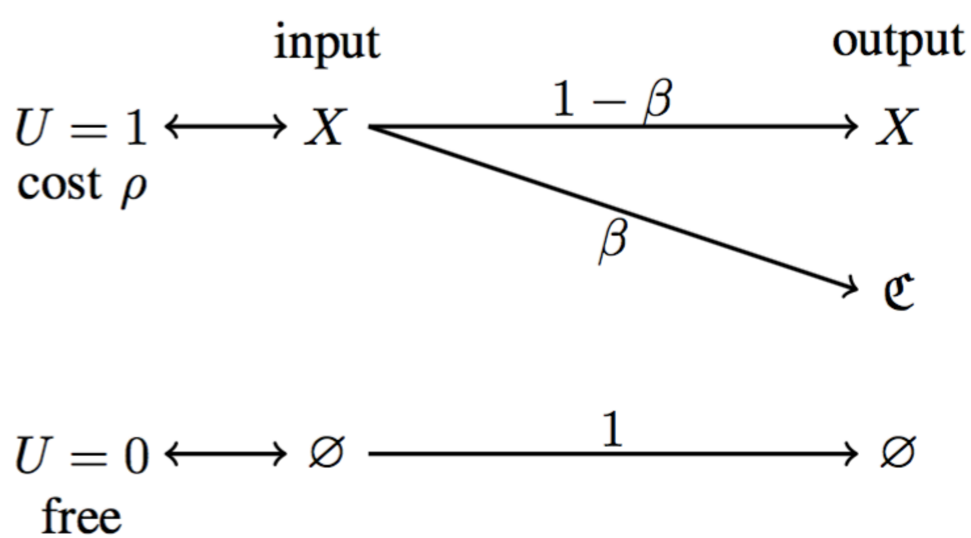
Minimize:

$$\mathcal{J}(a, b, \hat{x}_{\varnothing}, \hat{x}_{\mathfrak{C}}) = \int_{[a,b]} (x - \hat{x}_{\varnothing})^2 f_X(x) dx + \int_{\bar{\mathbb{R}} \setminus [a,b]} [\beta (x - \hat{x}_{\mathfrak{C}})^2 + \rho] f_X(x) dx$$

binary quantization problem

Why is this problem relevant?

1. **New channel model** for networked control
2. Collisions cause the **lack of symmetry** of optimal thresholds
3. **Building block** for other decentralized estimation problems



Collision channel

1. The **packet-drop channel** is a special case when $\varnothing = \mathfrak{C}$
2. The **collision channel** is widely used in wireless communications

Existence of optimal thresholds

$$\mathcal{J}(a,b,\hat{x}_{\varnothing},\hat{x}_{\mathfrak{C}}) = \int_{[a,b]} (x-\hat{x}_{\varnothing})^2 f_X(x)dx + \int_{\bar{\mathbb{R}}\setminus[a,b]} [\beta(x-\hat{x}_{\mathfrak{C}})^2+\rho] f_X(x)dx$$

$$\begin{aligned} &\text{minimize} && \mathcal{J}(a,b,\hat{x}_{\varnothing},\hat{x}_{\mathfrak{C}}) \\ &\text{subject to} && a \leq b \end{aligned}$$

$$x \in [a^*,b^*] \Leftrightarrow (x-\hat{x}_{\varnothing})^2 \leq \beta(x-\hat{x}_{\mathfrak{C}})^2+\rho$$

necessary optimality condition

$$a(\hat{x}), b(\hat{x}) \triangleq \frac{1}{1-\beta}\left[(\hat{x}_{\varnothing}-\beta\hat{x}_{\mathfrak{C}}) \pm \sqrt{\beta(\hat{x}_{\varnothing}-\hat{x}_{\mathfrak{C}})^2+(1-\beta)\rho}\right] \qquad \hat{x} \triangleq (\hat{x}_{\varnothing},\hat{x}_{\mathfrak{C}})$$

Define a new cost: $\mathcal{J}_q(\hat{x}) \triangleq \mathcal{J}(a(\hat{x}),b(\hat{x}),\hat{x}_{\varnothing},\hat{x}_{\mathfrak{C}})$

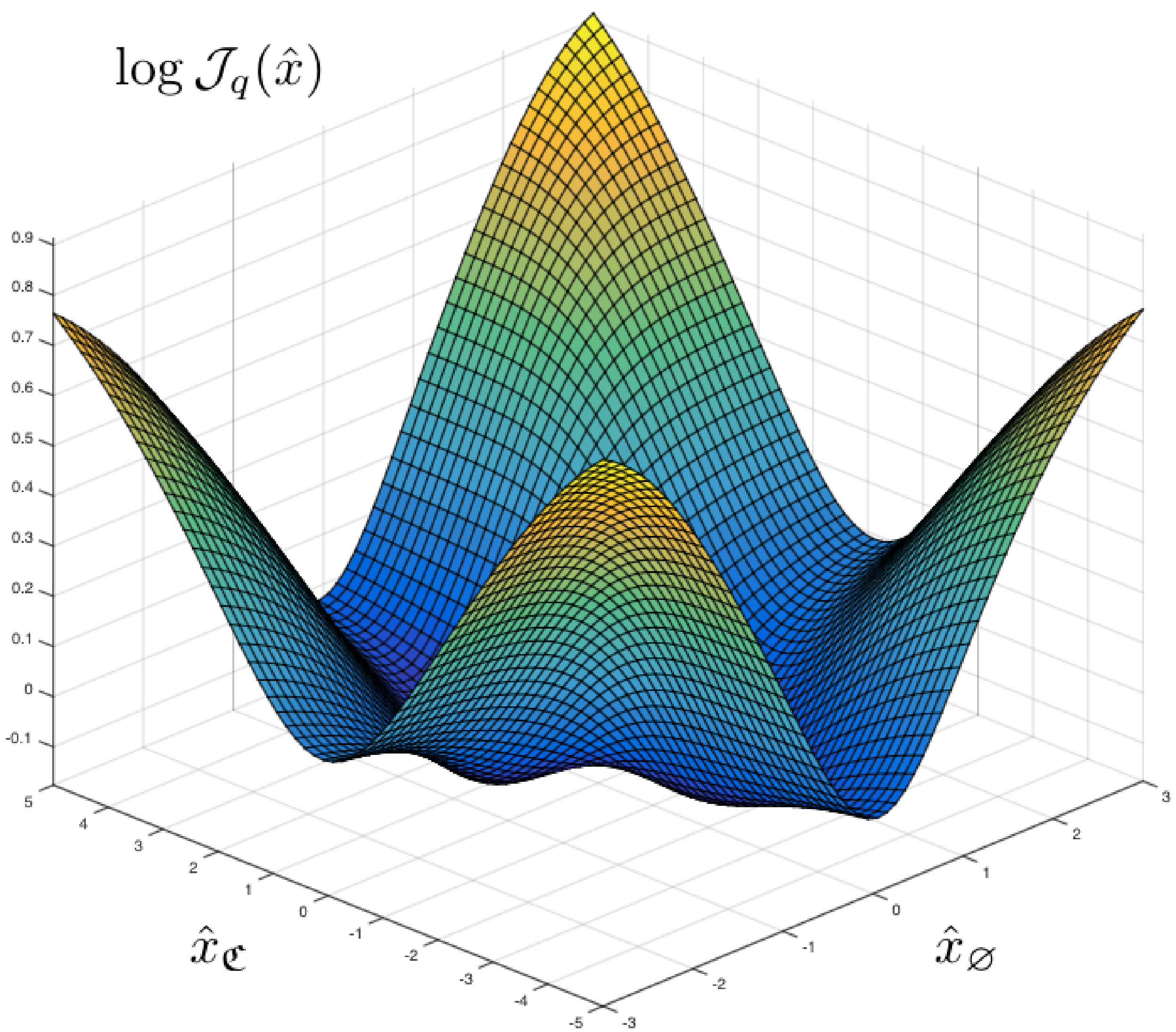
$$\text{minimize} \quad \mathcal{J}_q(\hat{x})$$

main mathematical object of this talk

Existence of optimal thresholds

Theorem 1:

Provided that $0 < \rho < +\infty$, the global minimizer of $\mathcal{J}_q(\hat{x})$ exists.



Coercivity:

$$\mathcal{J}_q(\hat{x}) \to +\infty \quad \text{as} \quad \|\hat{x}\| \to +\infty$$

does not hold!

Sketch of Proof:

1. The cost function is continuous on $\mathbb{R}^2$
2. There exists a point $\hat{x}^\star$ such that:

$$\mathcal{J}_q(\hat{x}^\star) \leq \mathcal{J}_q(\hat{x}) \quad \text{as} \quad \|\hat{x}\| \to +\infty$$

Existence of optimal thresholds

Step 1: There exists a symmetric policy that outperforms the ***always-transmit*** and ***never-transmit*** degenerate policies.

$$\mathcal{U}(x) \equiv 1 \qquad \mathcal{J}(-\infty, +\infty, \hat{x}_{\varnothing}, \hat{x}_{\mathfrak{C}}) = \beta(\sigma^2 + \hat{x}_{\mathfrak{C}}^2) + \rho \geq \beta\sigma^2 + \rho$$

$$\mathcal{U}(x) \equiv 0 \qquad \mathcal{J}(a, a, \hat{x}_{\varnothing}, \hat{x}_{\mathfrak{C}}) = \sigma^2 + \hat{x}_{\varnothing}^2 \geq \sigma^2$$

$$0 < \rho < +\infty \quad \text{implies} \quad \mathcal{J}\left(-\sqrt{\frac{\rho}{1-\beta}}, \sqrt{\frac{\rho}{1-\beta}}, 0, 0\right) < \min\{\sigma^2, \beta\sigma^2 + \rho\}$$

Step 2:

$$\mathcal{J}_q(\hat{x}) \to \begin{cases} +\infty \\ \beta(\sigma^2 + x_{\mathfrak{C}}^2) + \rho \\ \sigma^2 + x_{\varnothing}^2 \end{cases} \quad as \quad \|\hat{x}\| \to +\infty$$

cost of the ***always*** *and* ***never-transmit*** policies

■

The global minimum must exist

Asymmetry of optimal thresholds

First order optimality condition: $\nabla \mathcal{J}_q(\hat{x}^*) = 0$

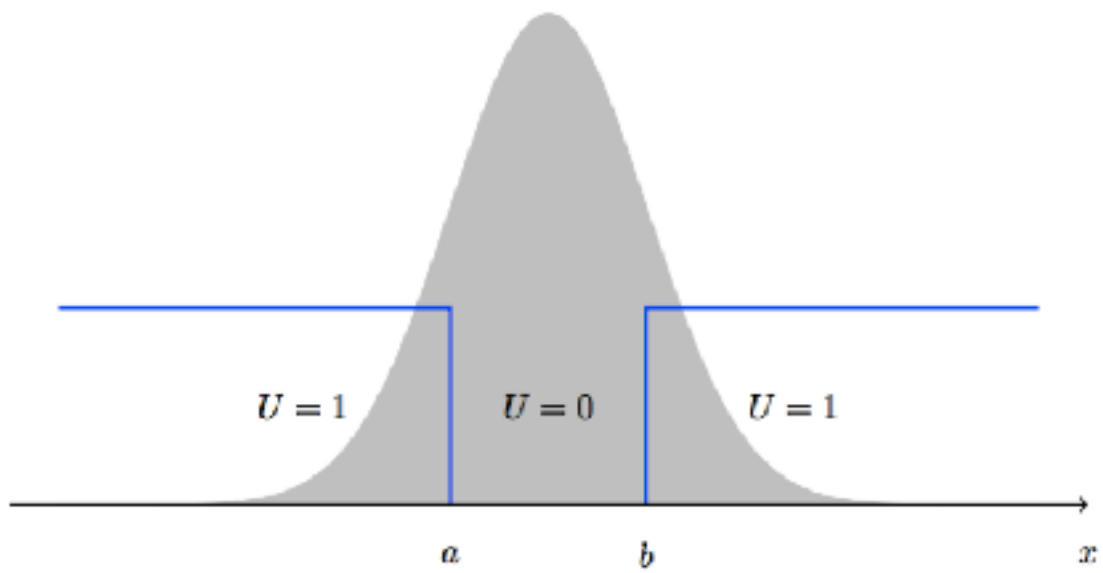
$$\hat{x}^*_{\varnothing} = \mathbb{E}\left[X | X \in [a(\hat{x}^*), b(\hat{x}^*)]\right]$$
$$\hat{x}^*_{\mathfrak{C}} = \mathbb{E}\left[X | X \notin [a(\hat{x}^*), b(\hat{x}^*)]\right]$$

Centroid condition

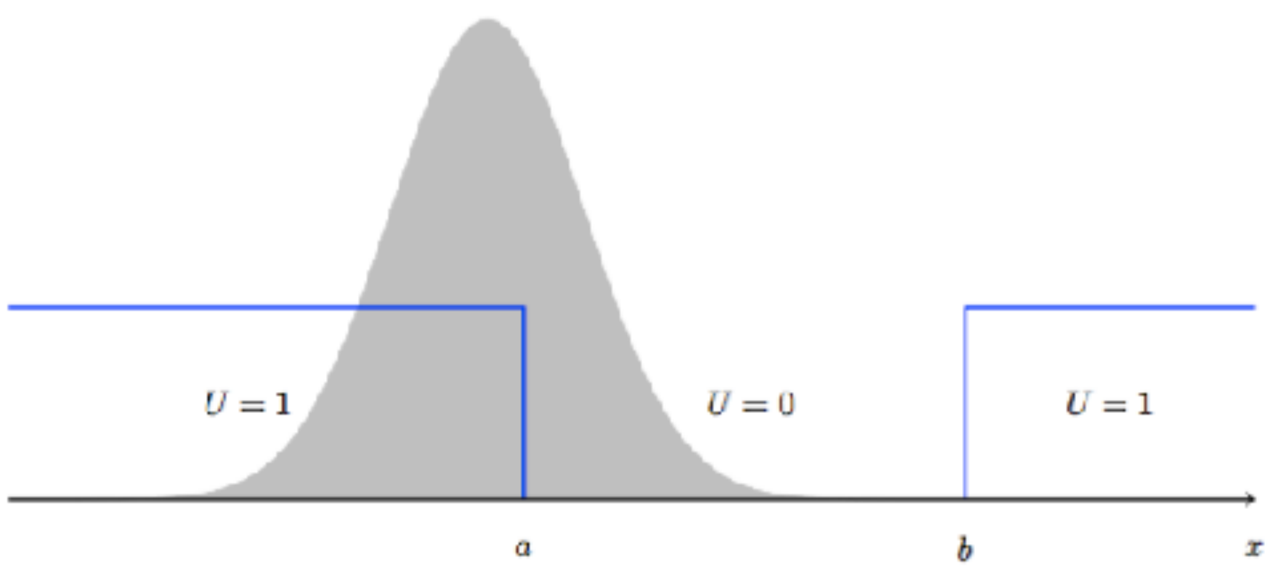
$\hat{x} = (0, 0)$ always satisfies this condition

corresponds to the best **symmetric threshold policy**

When are the optimal thresholds asymmetric?



Symmetric $a = -b$



Asymmetric $a \neq -b$

Asymmetry of optimal thresholds

Theorem 2:

If $\mathcal{G}(\beta,\rho) > \frac{1}{2}$ the optimal thresholds are asymmetric.

$$\mathcal{G}(\beta,\rho) \triangleq \frac{\mathcal{M}(\beta,\rho)}{2} + (1-(1-\beta)\mathcal{M}(\beta,\rho))\,\frac{\partial}{\partial\beta}\log\mathcal{M}(\beta,\rho)$$

related to $\det(\nabla^2 \mathcal{J}_q(0,0))$

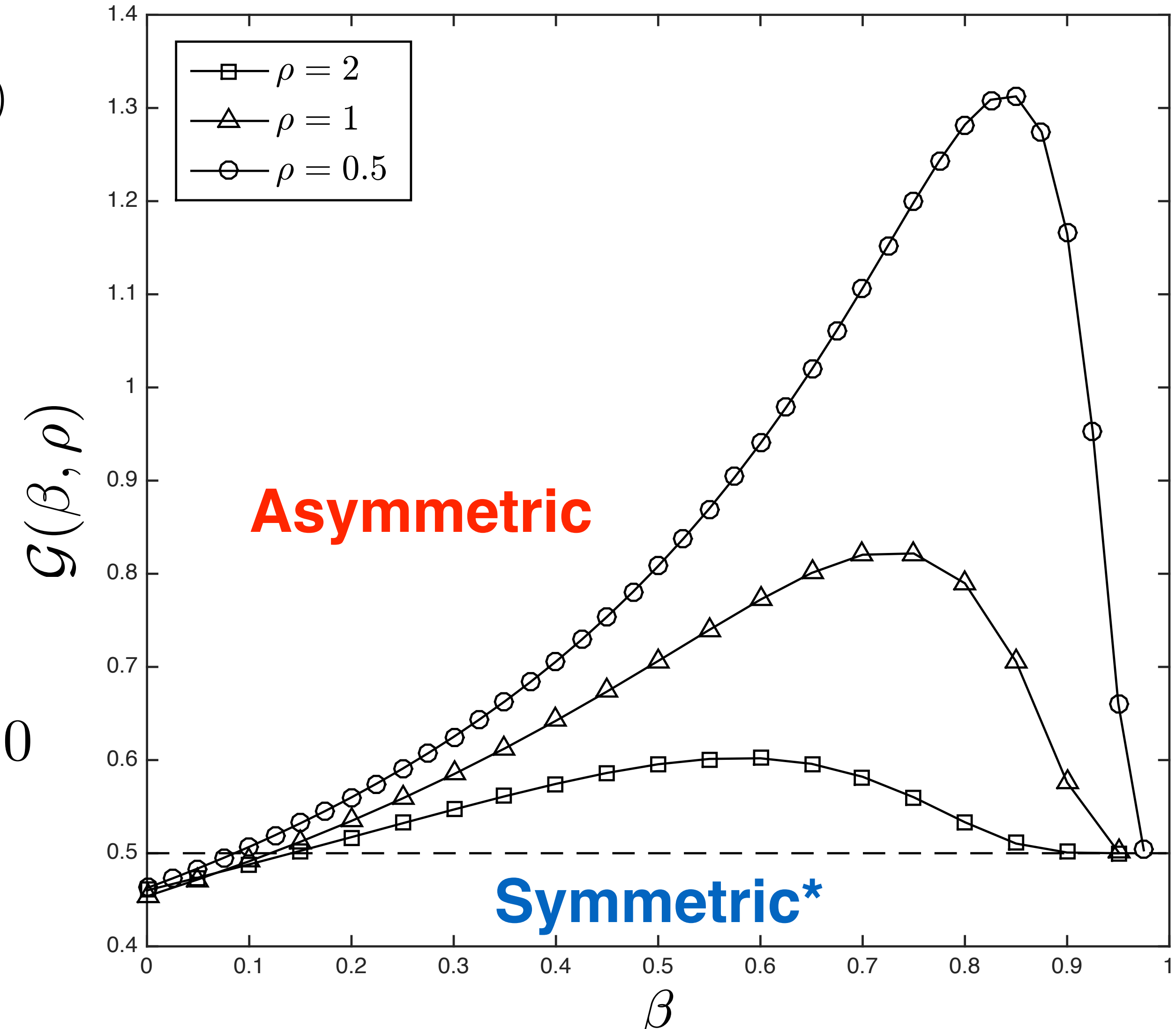
$$\mathcal{M}(\beta,\rho) \triangleq \int_{-\sqrt{\frac{\rho}{1-\beta}}}^{\sqrt{\frac{\rho}{1-\beta}}} f_X(x)dx$$

Sketch of Proof:

$\hat{x} = (0,0)$ is a local minimum then $\nabla^2 \mathcal{J}_q(0,0) \succeq 0$

$$\nabla^2 \mathcal{J}_q(0,0) \succeq 0 \Leftrightarrow \mathcal{G}(\beta,\rho) \leq \frac{1}{2}$$

$\blacksquare$



Modified Lloyd-Max and its convergence

$$\underset{\hat{x}\in\mathbb{R}^2}{\text{minimize}} \quad \mathcal{J}_q(\hat{x})$$

$$\nabla \mathcal{J}_q(\hat{x}) = 0 \qquad \longleftrightarrow \qquad \hat{x} = \mathcal{F}(\hat{x})$$

Lloyd's Map

$$\mathcal{F}(\hat{x}) \triangleq \begin{bmatrix} \mathbb{E}\left[X | X \in [a(\hat{x}), b(\hat{x})]\right] \\ \mathbb{E}\left[X | X \notin [a(\hat{x}), b(\hat{x})]\right] \end{bmatrix}$$

Modified Lloyd Max

$$\hat{x}^{(0)} \neq (0,0)$$
$$\hat{x}^{(k+1)} = \mathcal{F}(\hat{x}^{(k)}), \quad k = 0, 1, \ldots$$

Step 1 From $\hat{x}^{(k)}$ update the thresholds $a(\hat{x}^{(k)})$ and $b(\hat{x}^{(k)})$

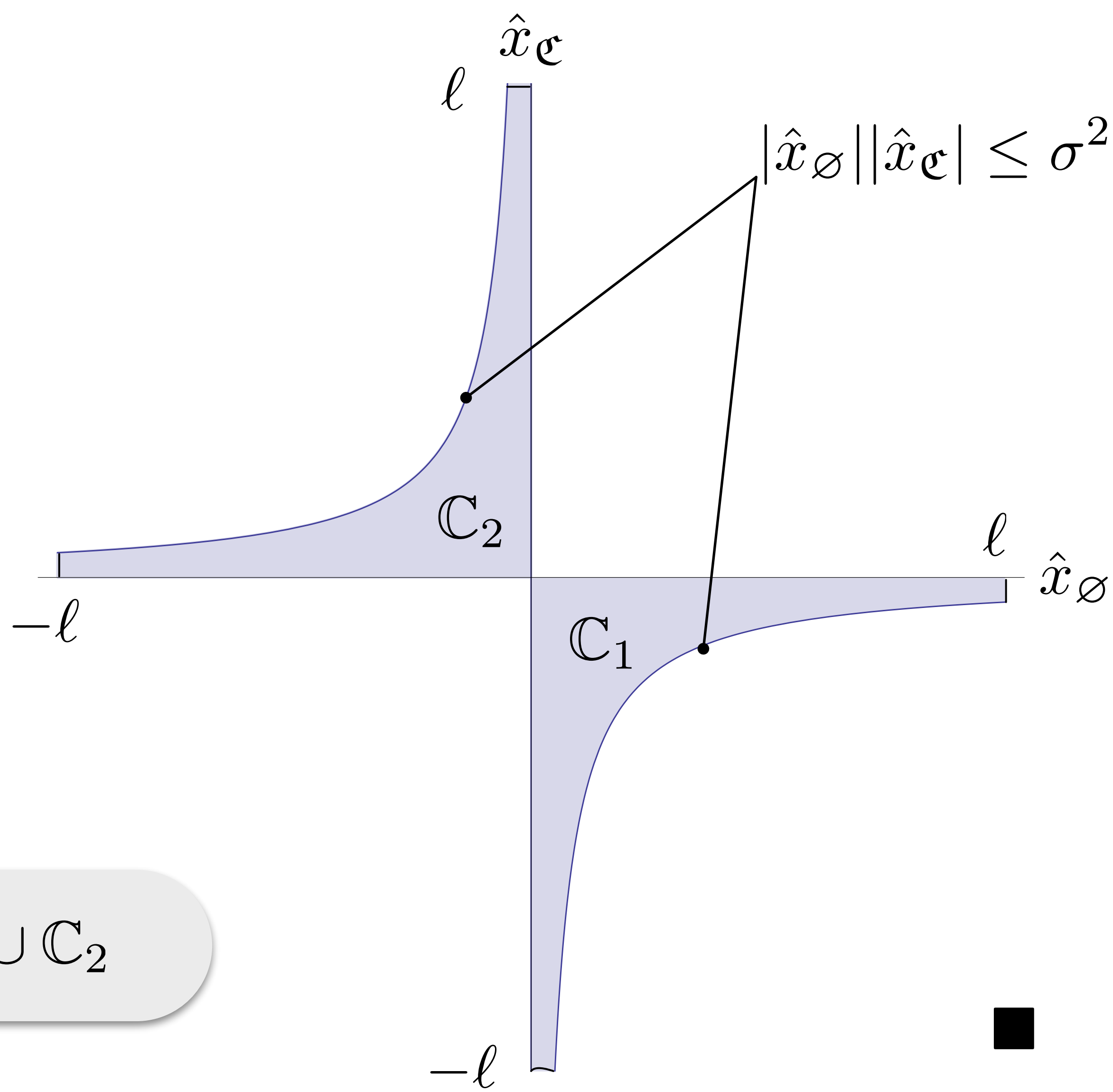
Step 2 Compute the centroids of the new quantization regions

Modified Lloyd-Max and its convergence

Theorem 3:

The modified Lloyd-Max algorithm is globally convergent to a critical point of $\mathcal{J}_q(\hat{x})$.

Sketch of Proof:



- Based on a result by **Qiang Du** - SIAM J. Num. Analysis '06
- Find a compact set $\mathbb{C}$ that contains all the critical points of $\mathcal{J}_q(\hat{x})$
- Show that $\mathcal{F}(\mathbb{C}) \subset \mathbb{C}$

$$\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$$

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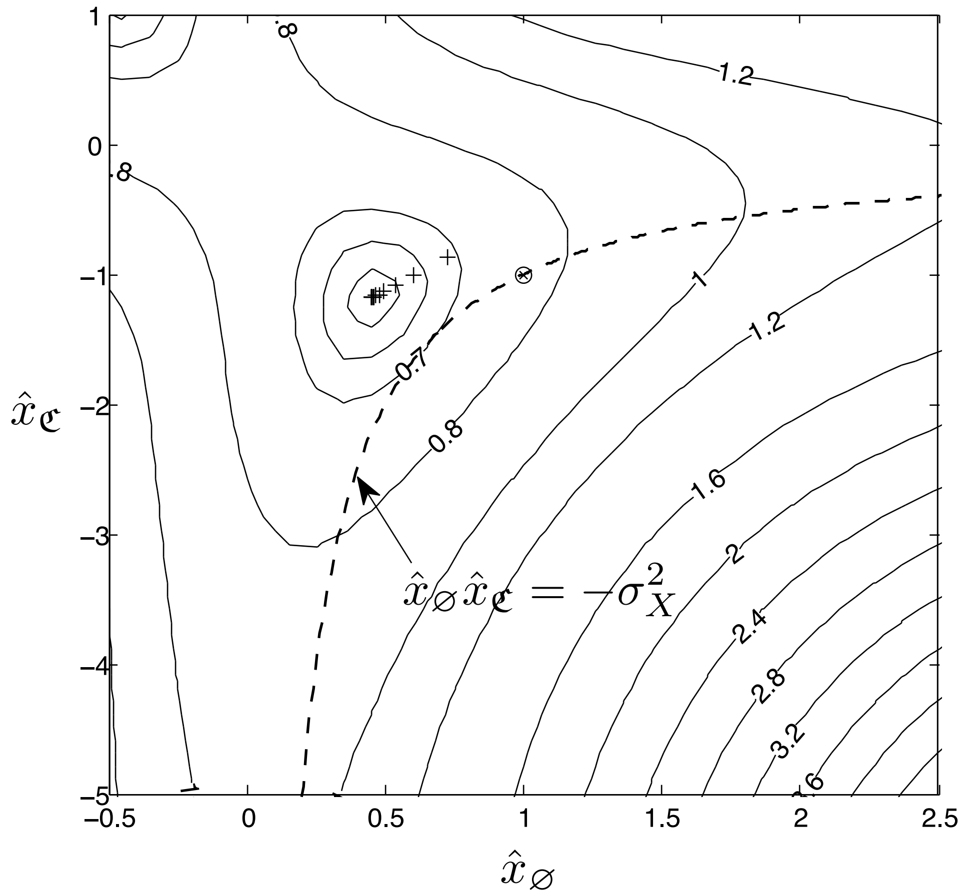
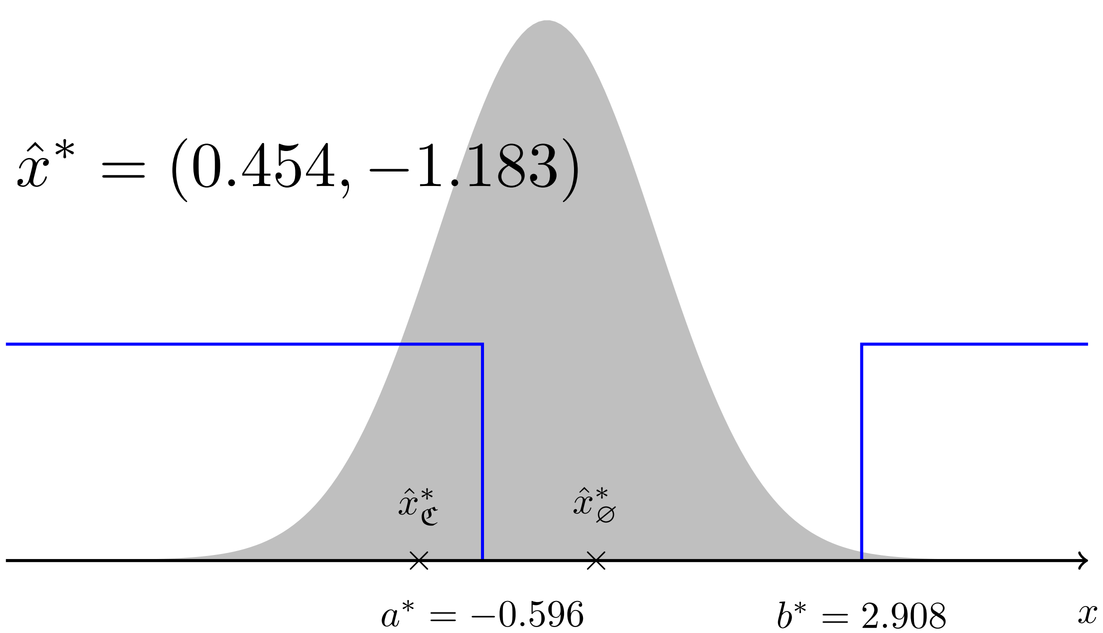
Examples

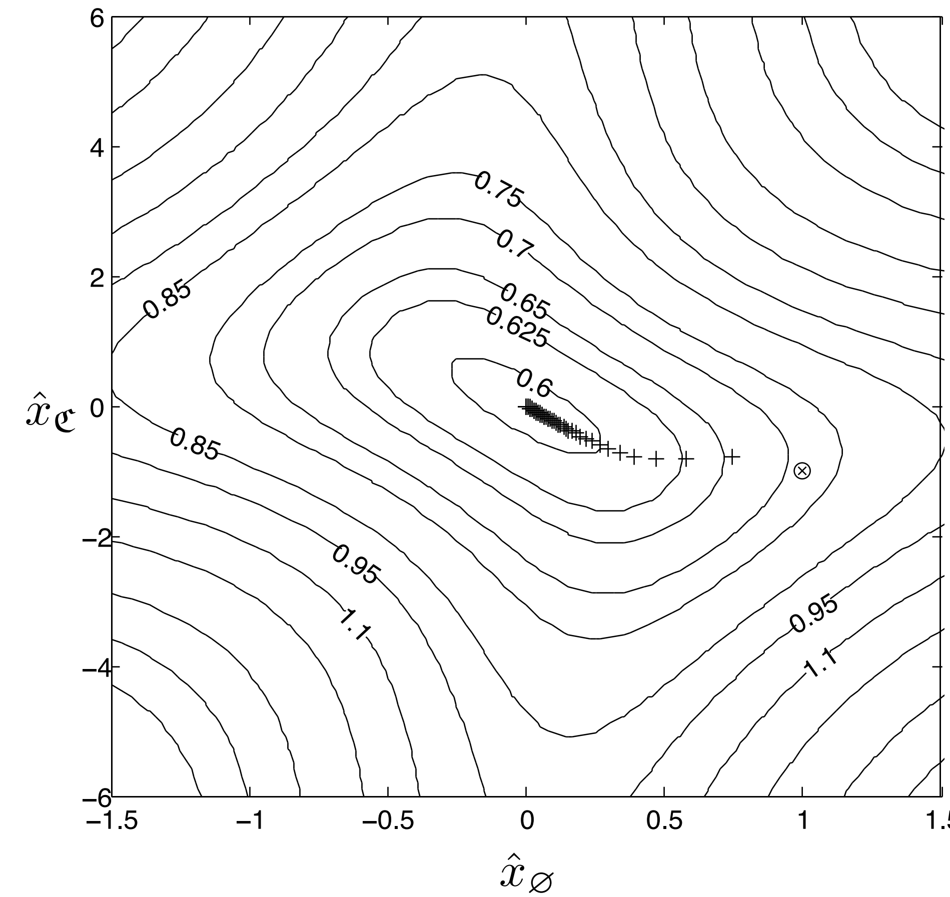
$X \sim \mathcal{N}(0,1)$, $\beta = 0.3$ and $\rho = 1$



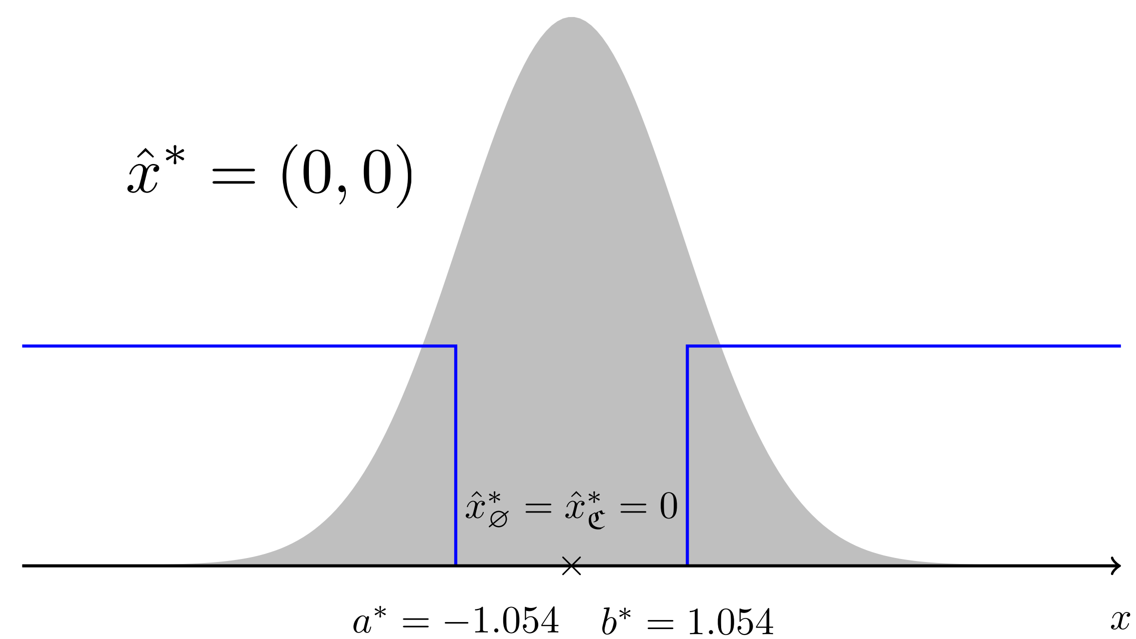
$\hat{x}^* = (0.454, -1.183)$



$X \sim \mathcal{N}(0,1)$, $\beta = 0.1$ and $\rho = 1$



$\hat{x}^* = (0, 0)$



Conclusion and future work

- **Existence** of optimal thresholds
- **Sufficient condition** for the **asymmetry** of optimal thresholds
- **Global convergence** of the Modified Lloyd-Max algorithm

- Solving the **sequential case** with channel output feedback
- Proving the convergence to a **global minimum**
- **Control** over the collision channel